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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Volume XXXIX Number 3

Edited by William David Reeve

Tenth Year Geometry for all American Youth

By SAMUEL WELKOWITZ

James Madison High School, Brooklyn, N. Y.

THAT there is need, indeed a clamoring for change in the mathematical fare offered to high school pupils in the tenth year is evidenced by: (1) the numerous teacher and supervisor meetings devoted to a discussion of the difficulties and problems of teaching plane geometry; (2) the gradual but steady drop, prior to the onset of the war, of the registers of mathematics courses, beyond the ninth year; (3) the rather doubtful results obtained by the pupils on Regents and other uniform examinations; (4) the increasing number of articles in current periodicals on the war and post-war slant of our mathematics courses; (5) frank statements of conscientious teachers of geometry.

In the forward looking brochure "Guiding Principles of Curriculum Development" (published by the Board of Education of New York City) we find the following pertinent remarks: "The broad purposes of education may be achieved through the use of a wide variety of activities and subject matter carried on both in and out of school. The learning experiences selected to constitute the curriculum should be chosen for any group of pupils and teachers working together from among the many possible activities on the basis of their suitability to the purposes, needs, abilities, and limitations of the group and the individuals who comprise the group. They should be selected also in terms of significant changes in social structure and governmental controls and services" (page 16). Also from the same source: "The curriculum is not static. It must always be open to revision. This is particularly true in times of war, public emergency and industrial changes. It should be possible to set up new courses or new procedures to meet special needs," (page 12).

The war intensified the feeling that something should be done to bring the tenth year mathematics course, as well as other courses in mathematics, into line with the war effort. It is universally admitted that our subject offers exceptional opportunities to help train and educate our pupils to play a more effective role in the war—on the fighting line, on the assembly line, on the civilian front—as well as during the reconstruction period in which we now find ourselves. Even if some sacrifices of traditional subject matter have to be made—and it is one of the purposes of this paper to show not only that there will be no ignoring or weakening of the desirable objectives but rather a strengthening and reviving of these objectives as a result of the proposed revisions —this procedure is not only justifiable but imperative.

To those who feel that any alteration

of the "Elements" is a desecration of that remarkable contribution of Euclid to the field of mathematics as well as to human thinking, we counter by asking whether it is not far more advisable to have the large majority of pupils appreciate some of the logical beauty of the subject than to have a constantly decreasing minority of them exposed to the Elements in "unblemished" but incomprehensible form. There is really no need of laboring this point. Numerous changes in content, order and methodology have been widely accepted and have resulted in the improvement of the teaching of geometry.

The traditionalists may and often do contend that many have survived and benefited from the present and older courses and have gone ahead to do good work in mathematics at college. Their attention may be directed to the many casualties, nay mortalities, which have accompanied and very frequently hampered the process. The subject of geometry should not be employed as a sifting or screening device. I feel that it contains so much that is valuable to all our secondary school students. Professor Wm. Hart states (School Science and Mathematics, Feb. 1943, page 115): "We must endeavor to discard the label college preparatory mathematics and justify substantial mathematics for all students of proper intelligence. When the substantial courses are well organized as to content—the subject matter can be defended as more important for some of the students who do not go to college than for others." In the recent book "Education for All American Youth" prepared by the Educational Policies Commission of the National Education Association we may find the following: "All American Youth are now living in the American culture—All require an understanding of the main elements in that culture—All American Youth have the capacity to think rationally, all need to develop this capacity, and with it, an appreciation of the significance of truth as arrived at by the rational process—Schools should be dedicated to the proposition that every youth in these United States—regardless of sex, economic status, geographical location, or race should experience a broad and balanced education which will—stimulate intellectual curiosity, engender satisfaction in intellectual achievement and cultivate the ability to think rationally" (pages 16, 17, 21). Thus, it would seem to be an educational blunder to restrict the cultural heritage of geometry only to the college preparatory or the so-called academic-minded pupil. However, these remarks are not intended to support the retention of the present course, which I believe is in a large measure responsible for its gradual rejection as one of the humanities of value to all our youth.

What are the failings of the present course in geometry? I shall base my criticisms of the course as it is widely practiced and as it is meagerly described in the New York State Syllabus in Plane Geometry. The State Syllabus disposes of the matter in one and a half printed pages. It prides itself (in the latest reprint) on encouraging "the development of the facility of independent thinking" (by reducing the number of required proofs from 38 to 32 theorems). And to show what little faith it has in the use of deduction it states, "The pupil will not be required to give proofs of constructions on the examination unless such proofs are specifically called for in the questions" (and no such proofs are ever called for). "To this end it is recommended that most of the propositions in the syllabus (about 100 of them are listed) that are not included in this minimum list be taken up formally or informally in class as the best means of developing a technic and skill in solving the originals." Note the great stress laid on the ability to solve originals (the traditional kind, of course). One of the common practices is to devote a considerable amount of time on the development of and drill on proofs of the 32 theorems on the minimum list. A great

premium, as a result, is placed on sheer memory with no corresponding beneficial effects such as ability to apply the theorems in the traditional originals or in the meaningful applications, or the appreciation of the nature of proof or of the logical structure of the subject.

There is an inordinate amount of attention given to constructions which have no other value than to provide practice in the use of the classical instruments (the unmarked ruler and compasses). Very feeble attempts if any are made to show how some of these constructions are used in shop constructions, lay-outs, navigation and design. No explanation is given for this strange restriction to the two instruments, when one may see so many other instruments being used in the drawing room, carpentry and machine shop. No effort is made to show the relation between the useless exercises in transformation of figures to the famous circle squaring problem. The deductive explanations of the correctness of most of the constructions are neglected or ignored.

Measurement and experimentation are discouraged by pointing to their unreliability as a method of discovering knowledge and, as a result, pupils ar given a wrong notion of the scientific method. A more critical study of the nature of deduction is discouraged by labeling such attempts as injecting pure logic and philosophy and other abstractions into a concrete subject (yet these same people discourage attempts to exhibit more clearly the empirical aspects of the subject and insist on limiting the study to that of "pure" geometry).

In short it is about time that we shifted the emphasis from developing "power"(?) in solving originals to clarifying the role of deductive geometry in explaining some aspects of our culture and civilization (e.g.—its relation to the scientific method and its application in more significant situations) rather than its use in the mere manipulation of axioms and theorems for its own sake or perhaps as a game. We should shift from development of the skill to reproduce a certain number of required proofs to development of a better appreciation of the nature of proof. Not only have so many pupils in the tenth year level not been able to cope successfully with the traditional objectives, indicated above, but they have been denied the opportunity (because of the lack of time resulting from the attempt to acquire the aforementioned "power"), to see the more valuable applications of deduction and to understand even partially the meaning of a postulational system.

In the light of the above, how valid or consistent are the following statements to be found in the Preliminary Bulletin of 1942 issued by the New York State Department of Education?

"These studies (made in various military training schools) reveal a striking agreement—that the present emergency does nor call for any radical changes in the content of our present syllabuses in mathematics not for a redefining of our accepted objectives in the teaching of the subject.—The emergency however, does call for the teaching of sound, basic mathematics in such a way that pupils will understand it, retain it and will be able to apply it. It would be a grave mistake to reduce materially the amount of academic mathematics that we are now offering in our regular courses. We should seek rather to redirect the emphasis along those lines which are most pertinent to our immediate problems without sacrificing any of the permanent values of mathematics. Nevertheless, the secondary school can increase its effectiveness in this respect (i.e. the introduction of war problems) by the introduction of illustrative materials and practical problems based on the present mathematics program without distorting its framework and without sacrificing its basic values for general educational purposes." Nowhere can we find a definition or clarification of the terms "basic values or permanent values" of mathematics or even of geometry. The

rest of this article will be devoted to my interpretation of these terms which are bandied about so loosely or, worse, which are used to serve as halos for ideas sorely in need of defense.

Professor Dorwart, after discussing the high place which mathematics once held in the schools and the poor results in mathematical education shown by recent army reports, said (THE MATHEMATICS TEACHER, Feb. 1945, page 83): "I now propose the thesis that, once the requirements were withdrawn students ceased to study mathematics principally because *they did not recognize the fundamental role that it plays in modern civilization.*" In a review of Professor Barzun's book "Teacher in America" the reviewer (Times Book Section, Feb. 25, 1945) commenting on the chapter on "The Ivory Lab" states: "The majority of the scientists refuse to admit that they have any function except to propagate more research men. The occasional scientist who regards his specialty as one of the humanities is likely to be considered a little daft by his colleagues. Until the scientists come down out of their ivory labs and join in the common enterprise, the majority of the college students will continue to be scientific ignoramuses." How much more apropos are these reflections on the level of secondary education!

The concept of the complementary roles of the practical and theoretical must displace the rather prevalent notion or at least displace the common practice that tries to show that the two aspects of science are conflicting and that one of them, usually the practical, is to be deemed inferior. Sir Percy Nunn exploded this harmful myth in his book on the Teaching of Algebra (p. 16)—"Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another . . . . The history of mathematics is a tale of ever-widening development of both these sides. From its dim beginnings by the Euphrates and the Nile mathematics has been on the one hand a means by which man has constantly increased his understanding of his environment and his power of manipulating it, and on the other hand, a body of pure ideas, slowly growing and consolidating into a noble rational structure.—But these different currents of progress must not be thought of as independent streams. One has never existed and probably will never exist apart from the other. The view that they represent wholly different forms of intellectual activity is partial, unhistorical and unphilosophical. A more serious charge against it is that it has produced an infinite amount of harm in the teaching of mathematics.—Nor should we add a clause to safeguard the interests of those who are to enter the mathematical profession. The treatment of the subject presented (above) by our principle is precisely the one which best supplies their special needs."

Professor Nagel in an article in American Philosophy Today and Tomorrow (page 383) states: "The doctrine that the sciences or philosophy have as their subject matter a special realm of reality, sometimes alleged to be inferior, sometimes superior to the experimental world of common-day life does not lack supporters today. Eminent scientists rise from time to time to assure us that there is a 'real' world to which we may gain entrance, via some recent theory, which is free from the precarious, qualitative traits of every day nature. Or philosophers, not to be outdone in nonsense, declare with assurance that the sciences deal with mere abstractions existing nowhere except in the mind of the scientist, so that 'reality' must be 'known' in some other way—religious experience, art, or social participation." Professor Hook (in the same book page 216) states in this connection: "But the very claim to be dealing with a subject matter at no point infected with the 'evil' of existence is at the source of many of the difficulties and paradoxes of

contemporary logic and mathematics." Professor Hook quotes this from John Stuart Mill's autobiography: "The notion that truths external to the mind may be known by intuition or consciousness, independently of observation or experience, is, I am persuaded, in these times, the great intellectual support of false doctrines and bad intuitions.—And the chief strength of this false philosophy in morals, politics and religion, lies in the appeal which it is accustomed to make to the evidence of mathematics and of the cognate branches of physical science." Likewise Will Durant, in the same book, says (page 124) "Understanding the word in this double sense, I should say that sensation rather than reason or intuition, is the least unreliable of our tests of truth. Certainly the senses may deceive us, but we find this out only by other senses.—The most subtle reasoning is in the end nothing but a tentative prediction that under given circumstances certain sensations will be received. Science is the organization of sensory experience into formulas susceptible of sensory verification. Truth is permanently consistent sensation."

This dichotomous division of geometry into pure and applied must be re-examined. The segregation of pupils into two groups—those who are to be given applied mathematics and those who are to be given pure mathematics—is, I believe, educationally and socially indefensible. If differentiation of pupils is necessary it should be based on the *amount* not the *kind* of mathematics that pupils can absorb. All pupils should be exposed to both pure and applied mathematics. Differentiation can also be based on the fields of application—such as science, art, technology, shop, recreation. If some pupils cannot absorb the pure mathematics that is offered them, may it not in a large part be due to our inability to furnish the requisite background of applied mathematics and to motivate and to clarify this pure mathematics? The more capable pupils can be gotten to apply the same pure mathematics to more difficult practical situations and, perhaps, thereby, stimulate a desire for more pure mathematics.

Hence I cannot subscribe completely to all the views expressed by Professor Breslich in 1938 (MATHEMATICS TEACHER, Nov. 1938, p. 314). He classified pupils into two groups. One "a large group which has been increasing in the last decades and in which are found those who dislike deductive geometry because they do not understand it, a second group, decreasing from year to year, which consists of pupils who study geometry because they enjoy it, and in many cases also because they plan to prepare for future work, in which a knowledge of geometry is essential.—Deductive geometry as it is now organized serves the second group in an admirable way. Those of the first group who dislike it may take it or leave it. Perhaps the time has come to build a second course in geometry which will appeal to the interests and serve the needs of the first group as well as Euclid's geometry has served the second for centuries. Such a course should be practical above anything else, and those propositions that have the best applications should be the ones to be emphasized. The method of study should be inductive and experimental, although some deductive reasoning should have a place. Geometric constructions, drawing, and designing should receive much attention.—The pupils should make frequent excursions into three-dimensional space. Most of the pupils taking such a course would probably be of the non-collegiate type, but many others would take it even though the colleges could not recognize the course for credit."

It is commonly accepted that one of the main objectives of the course in geometry is to develop an appreciation of geometry as an example of a science. Unfortunately the methods generally employed do not achieve this aim. Burdening the pupil with a list of definitions and postulates and then telling him "to build his geometry

very much as the workman builds a house, by first gathering the stones for the foundation. After the foundation is laid he adds one story at a time until the dwelling is complete." (Breslich—above mentioned article). Is this a fair description of the scientific method? Note, by way of contrast, what is advocated in "Education for all American Youth" (page 131)! "Not only scientific facts are taught (referring to the school of the future) but also the scientific methods by which facts are discovered. Every elementary classroom is, therefore, used as a laboratory for simple experiments; and in secondary school, experimentation is carried much further. —By constantly practicing scientific inquiry, students develop a knowledge of experimental methods, an understanding of the nature of proof, and a respect for truth arrived at by rational processes, which they could hardly gain in any other way.—He will know that most scientific advances have depended upon precise measurement and accurate calculation and that mathematics is indispensable to scientific inquiry." Breslich in this connection states (above article p. 312)— "However induction should have a place in demonstrative geometry. It should not be entirely dropped, because it is the method of experimentation and generalization. By applying both deductive and inductive methods to geometric facts, the pupils will be led to an appreciation and understanding of what is meant by logical thinking.—Many pupils have more faith in these (intuitive and experimental) methods than in deductive proofs. They will throw aside a perfectly good proof any time if the facts do not seem to conform with experience. It is good teaching of high school geometry to prove experimentally those theorems with which pupils in general are known to have difficulty. The deductive proof should follow when there is evidence that the meaning of the theorem is understood." In my opinion Breslich has not shown clearly how to exhibit geometry as an elementary illustration of the scientific method. He has failed to point to the complementary roles of experimentation and deduction.

In the tenth year the importance and techniques of experimentation should be brought to the attention of the pupils in a more conscious and organized manner. Such ideas as the following should be introduced: the proper use of measuring instruments, the notion of fair sampling, the value of group cooperation in experimentation, and most important of all, that demonstration is a systematic method of explaining and predicting results obtained in one experiment by means of the results obtained in preceding experiments. The interrelationship of experimentation and deduction is one of the chief characteristics of the scientific method. This juxtaposing of experiment and deduction will also show more clearly to our pupils the beauty, economy, and power of the deductive method. At times experiment may be used to instil greater faith in results obtained by deduction. Permit me to cite the example of the pupil who gave what appeared to be a perfect recitation on the proof of: "Each angle of an equilateral triangle equals 60°" and then turned to the teacher and asked "Is this true for every equilateral triangle?" In this instance the pupil demanded "proof by conviction" and was not contented with "conviction by proof."

Pupils have not mastered the above notions unless they can answer intelligently such questions as "What is the purpose of proving a principle by deduction after you have discovered it by experiment?"; "Explain the differences between the methods of experiment and deduction in the following respects: (a) the number of samples (diagrams) that should be used (b) the accuracy of construction of diagrams (c) the extent to which previously established principles are used." Pupils should at the close of the course be able to appreciate such a statement (or its equivalent) as, "The scientific method is the interplay of the intellect and the

senses." The power of deduction should not be ascribed, as purists and a priorist often do, to the workings of the mind alone but rather the impact of the mind upon the real world as revealed by observation and experiment; that the genius of Newton operated upon the observations and tabulations of Tycho Brahe and Kepler. Abstract or philosophical references to the scientific method are inadequate. It is essential to provide significant and, if possible, novel and unexpected, settings for experimentation. For example "Determine whether an angle bisector of a triangle is also a median," or "Determine whether the two angle trisectors of an angle of a triangle are equal." These will provide opportunities for careful class planning of the experiment, the selection of appropriate samples, practice in the use of the ruler and protractor, the making of tentative judgments and finally deducing the conclusion discovered.

One of the cardinal aims of mathematical instruction advocated in the National Committee Report on the Reorganization of Mathematics (1923) was "to develop those powers of understanding and analyzing relations of quantity and space which are necessary to an insight into and control over our environment." In Education for All American Youth we find: "You will also see (in the school of the future) a great deal of learning of facts and principles which will long be remembered and applied because things are being learned in connection with *their uses in life.*" This does not seem to jibe with Breslich's comment: "When an adult is asked to state reasons why he enjoyed the study of geometry he usually fails to mention the practical applications among his reasons." Dean Ingraham in an article on Mathematics and Science in a Liberal Education (School Science and Mathematics, Feb. 1945, p. 131) states: "I am, therefore, not pleading for a life of a scientist to the exclusion of all else but for the enrichment which science can bring to a life which is also to be enriched by the study of the humanities. I think, therefore, one should consciously consider in teaching either mathematics or science that broad relationship of man and his environment should be revealed to the student and the plain pleasure that even a non-scientific student can get from the study as well as any technical efficiency that may be attained." Professor Sanders in the National Mathematics Magazine states "For, even in prewar times there had been for many years a steadily growing public emphasis upon applied mathematics, rather than upon the logical or cultural aspects of the science. In the light of this definite trend, a trend not rooted in any war, it could well be that the post-war school effort should first be directed to discovering the mathematical aids or needs of all the major peace-time industrial enterprises. Cooperative programs initiated between industry and the schools would then have sounder foundations. Who shall say that the cultures of mathematics would be impaired by being stemmed in its utilities?"

The High School Division of the New York Schools (in its circular No. 45 of 1942) directed that "The Chairmen of the Mathematics Departments are to introduce immediately practical and significant applications of the principles taught in their courses." Professor Christofferson in an article on "The Mathematics Basic to Navigation" (School Science and Mathematics, March 1943, page 263) states: "Mathematics can be made concrete and simple. It can be made meaningful and functional. It is and ever will be widely useful in a variety of activities, not only in navigation, but in the machine shops, in the factories, stores, highways, science laboratories and even in the solution of many social and economic problems. As mathematics teachers it is up to us now to utilize the tremendous impetus that has been given the teaching of mathematics that our subject may function adequately, completely and understandingly in the lives of American Youth.—It is easy to

see that these typical air navigation problems involve some beautiful mathematics and seem so fundamental in their importance that it would not seem difficult to justify their inclusion in the education of every boy and girl, even though they may not be the navigators of airplanes."

Dean Freeman of the School of Education at the University of California recently said, "The mathematics for the millions is that which gives clearer and more effective ways of thinking about the real world of experience because it has been developed out of this world of experience." It is about time that we banished the classic Greek disdain of the practical and encouraged textbook writers to advance the position of such applications to a more prominent place in their books.

The role of applications in the teaching of geometry has hitherto been a curious one. They were used mainly as a motivating device with which to whet the appetite of the student for the study of a new theorem. But once the thorem was taught it was used, in the main, to prove more and more originals which rarely had any connection with extra-geometrical situations. One of the major objectives of the teaching of geometry should be the constant tying up of geometric principles with applications in the practical arts, the natural sciences, industry, design etc. The importance of instruments and practical appliances in modern civilization is too obvious to need any elaboration. Most of them involve some laws of geometry and some of them are nothing but the very embodiment of some theorems in geometry. Practice in the analysis and discovery of those laws of mathematics which are the basis of these instruments will not only afford exercise in doing real originals, but will also give meaning and reality to a process that was traditionally barren and stilted. Instruments such as the T-square, tri-square, center-square, parallel ruler, protractor, proportional-dividers, pantograph, angle mirror, sextant, the plane table and others, should be studied, analyzed and treated as an integral part of the course in geometry. The navigator, the surveyor, the artisan and others who rely on instruments to obtain data also use, consciously or unconsciously, geometrical laws in the pursuit of their professional activities which consist in the main of making deductions from these data (e.g. indirect measurement). The analysis and discovery of these laws should be an important part of the course in geometry. If necessary, we should take the time requisite for an acquaintance with some of the elements in the related fields so as to make the geometric applications more meaningful. The making of models, designs, and some simpler instruments will also help to lend a feeling of reality and meaningfulness to many of the theorems and constructions of geometry.

The traditional interpretation of "construction" in geometry tends to handicap the pupil of the tenth year in many ways. Because of these restrictions and limitations—which might properly be called the idolatrous adulation of the classical restrictions, the historical backgrounds of which are not clearly authenticated and whose mathematical implications are far beyond this pupil's scope—the manual, or manipulative possibilities of the subject are needlessly circumscribed. Why should suspicion be cast on the value of many of the practical instruments used by the draughtsman, carpenter, engineer, machinist etc? In many instances these instruments (the marked ruler, the protractor, the square edge, the T-square) 30°–60° and 45° right triangles, the proportional dividers, the parallel rulers) help perform constructions in a simpler and in just as accurate a way as are obtained by the exclusive use of the straight edge and compasses. This stubborn adherence to the ivory-tower procedure, while it may have some interesting mathematical sidelights—some of which may be called to the attention of our students—denies the

pupil an acquaintance with real life practices. Certainly the expert use of the protractor and the ruler marked off to $\frac{1}{16}''$ or $\frac{1}{32}''$ is of far greater importance to a person growing up in our civilization than the use of the traditional instruments with their obvious limitations.

The constructions performed with the non-Platonic (?) instruments afford in addition, practice in deduction by trying to explain why they "work." The greater variety of problems that can be offered as a consequence of this extension (e.g. navigation, scale drawing, copying of designs) are more concrete, practical and appealing to the average pupil. Those traditional theoretical constructions, for which no practical analogies can be found, can be treated as interesting recreations, with far greater emphasis on their deductive than on their purely manipulative aspects. The purists should also be made aware of the fact that they are violating Euclid's restriction against drawing a circle with a compass-carried distance (pointed out by De Morgan). Professor Bussey in an article on "Geometric Constructions Without The Classical Restrictions to Ruler and Compasses" (American Mathematical Monthly, May 1936 pages 265–280) states: "High School students may well be told about the famous geometric problems of antiquity and that they cannot be solved with ruler and compass alone. But they should not be told that and nothing more. They might be told how easy it is to solve one or two of these famous problems by other simple means and thus dispel the prevalent notion that mathematicians have been unable to solve such simple problems as that of trisecting an angle. Students ought to be told more about what *can* be done and not exclusively what *cannot* be done with simple geometric instruments. If college teachers do not give their students more of this material than they have in the past, the next generation of high school teachers will be as fully bound by the classical tradition as their predecessors; and they will have missed a lot of interesting subject matter that is very definitely related to their job as teachers of elementary geometry."

The word construction should be construed in its literal and a broader sense. By employing a greater variety of instruments on such materials as paper, cardboard, wood, metal and plastics, the pupil can be encouraged to make designs, models of appliances and simple instruments.

Dean Ingraham in an article quoted from before (Mathematics and Science in a Liberal Education—School Science and Mathematics, Feb. 1945, page 131) also states: "I would place more emphasis on proofs than is currently the fashion. Please do not misunderstand me in this regard. I think there is little value in memorizing a proof in the step by step detail fashion that some students resort to. I feel that greater emphasis on the nature of proof and the understanding of proof is needed. Proofs of theorems indicate the inter-relationship between one portion of a theory and its general background. It is only through proof that the structural pattern is seen. It is only through proofs that the constructive imagination of a mathematician is understood. It is chiefly through the proof that the beauty of mathematics is revealed to the student."

We cannot safely boast that geometry, as taught at present in most places, achieves these objectives. Because of the inordinate burden which is placed upon the memory of the average student he can scarcely see the forest for the trees. He never gets around to appreciate the nature of proof because he is kept so busy "proving things." The present Regent's requirement to master thirty-two proofs does not help the pupil appreciate the notion of sequence. Most pupils are not mathematically mature enough to follow the complete logical thread. The New York City Standing Committee on Mathematics advocated: "It would seem to be a wise expenditure of time to set up a cross-

examination of each proof rendered in order to foster the spirit of confident rejoinder and to arouse an appreciation of the implications of hypothesis, conclusion, statements and reasons. The point to be stressed is not ground to be *covered* but ground to be *cultivated*. More emphasis should be placed on the nature and meaning of deductive reasoning or the process of drawing necessary conclusions from a given set of assumptions and its application in all life thinking. Less emphasis should be placed on the solving of originals and more attention should be given to the appreciation of the nature of reasoning and types of reasoning."

In line with the above the following guiding principles are recommended in selecting the propositions to be retained. Only those propositions should be retained for deductive proof or for factual knowledge or both which fulfill at least one of the following conditions: (1) They have many varied and interesting applications in the sciences, industry, shop, navigation and the arts of war (2) They form an indispensable link in the logical chain of reasoning. In the latter case it may sometimes be more desirable to assume the truth of the proposition on the basis of an informal proof or experiment. In an experimental course which is being carried on in a number of New York City high schools the following treatment of theorems is employed:

(a) *Theorems derived mainly for application purposes.* Their selection is based on their wide applicability or their indispensability in deriving theorems with wide applicability. These theorems are to be developed in class completely—formally, informally or experimentally. The testing of these proofs should not be in the traditional form (i.e. for mere reproduction) but rather in short answer or completion form. An attempt should be made to show sequence in small groups of theorems and to build up "family trees" occasionally. By far the greater emphasis should be placed on the applications rather than on the proofs; pupils should be permitted, nay encouraged, to refer to their textbooks in selecting their supporting principles; pupils will of their own accord memorize the statements of those principles used frequently.

(b) *Theorems for reproduction and analysis.* The requirement to reproduce a certain limited number of proofs may be justified on the following grounds: (1) They will serve as model forms for writing complete deductive proofs. (2) In this completed form it is possible to make a further study of and to test the concept of the nature of proof. (3) They will serve as illustrative cases of the main methods (e.g. triangle congruence, triangle similarity, parallel lines etc.). No more than about ten theorems are recommended for this purpose.

1. The sum of the angles of a triangle $=180°$. (This employs parallel line principles and serves as a good follow up of experiment by deduction.)

2. If two sides of a triangle are equal, the angles opposite them are equal. (This illustrates the use of the SAS theorem.)

3. If two angles of a triangle are equal the sides opposite them are equal. (This employs the SAA theorem and is a simple illustration of the converse.)

4. The opposite sides of a parallelogram are equal. (This employs the ASA theorem and parallel line facts.)

5. If the opposite sides of a quadrilateral are equal the figure is a parallelogram. (This employs the SSS theorem and is another good illustration of the converse.)

6. In a right triangle the altitude upon the hypotenuse is a mean proportional to the segments of the hypotenuse. (This employs one of the similar triangle principles.)

7. The area of a triangle equals $\frac{1}{2}$ its base times its altitude. (One of the easiest of the area theorems.)

8. A diameter perpendicular to a chord bisects the chord and its arcs. (This depends on the hypotenuse—arm theorem

and it has numerous converses.)

9. Tangents to a circle from the same external point are equal; and especially a corollary: The line bisecting the angle formed by two tangents passes through the center of the circle. (The basis of the center-square instrument.)

10. If 2 chords intersect the product of the segments of one chord is equal to the produce of the segments of the other chord. (It employs similar triangles to prove products of lines equal.)

11. If a tangent and a secant be drawn to a circle from an external point the tangent is a mean proportional between the secant and its external segment. (Another mean proportion theorem using angle measurement facts.)

Professor Reeve (THE MATHEMATICS TEACHER, Jan. 1943, page 16) says: "In geometry it will be necessary to omit about $\frac{2}{3}$ of the traditional propositions to be fully proved. The real purpose of logical geometry can better be secured by retaining only the necessary basal propositions."

The need for systematizing thinking should evolve slowly. It cannot be imposed and cannot well be learned in a void. Many pupils do not see the need for a complete formal proof at the beginning of the term. The deductive process is assimilated at a more gradual pace. The spirit rather than the form should be emphasized at the beginning. The stages of this development may follow this pattern: (1) The very informal stage where only one principle is required to justify a conclusion (e.g. numerical examples on vertical angles or on parallel lines). (2) The informal stage where two or three main principles are needed to justify a conclusion. Only the statement of these principles should be required at this stage. (3) The semi-formal stage where the pupil has to arrange the steps in proper sequence; the minor steps, use of auxiliary lines and definitions may be omitted. (4) Complete formal proof where all supporting steps are to be shown; axioms should be introduced gradually. A study of the nature of proof, the last stage, should constitute a part of each lesson, if possible, for the balance of the course.

After a gradual building up of a collection of principles in the first few weeks of the course, there should follow a study and then a classification of these principles into the categories of definitions, previously deduced propositions (theorems), previously accepted but undeduced propositions (axioms and those obtained by experiment only) and hypothesis. Pupils should be led to see that these are the only acceptable types of reasons. As drill on this unit pupils may be asked to indicate in a third column the category of each reason offered.

A further study of proofs (those appearing in the textbook and those built up by pupils) should include a test for the relevance of each reason. Very often a pupil states a principle which is one of the four acceptable kinds but which does not apply in the particular instance. For this purpose it would be advisable to paraphrase all reasons, if possible, in the "if . . . then" form. The pupil should be ready to support the "if" part with some preceding step (or steps) and the "then" part should be matched by the very same step. The relevancy of steps should also be tested.

Later in the course the pupils should be led to see that, throughout, only the following elements were employed in deduction: defined terms, undefined terms, undeduced propositions and deduced propositions, and they should be informed that all deductive sciences possess this property. Pupils should be required from time to time to give illustrations of the above and to classify any term or proposition in terms of these categories. Here again the direct method of teaching is necessary if some measurable results are to be obtained in the appreciation of postulational thinking.

A study of sequence within a given proof may be followed by asking such questions

as: "Which 2 steps may be interchanged?" "Which 2 steps may not be interchanged?" Sequence among theorems may be exhibited by grouping sets of theorems (e.g. parallel lines, sum of the angles of a triangle, sum of the angles of a polygon). Such questions as the following may be asked: "State 2 theorems upon which the proof of . . . is based." "State a theorem which may be derived from . . . ." "In proving a theorem on page 100 of your textbook the author could not use which of the following as a reason: an axiom, a theorem on page 150, a theorem on page 75, hypothesis, the theorem on page 100?"

The notions of converse, inverse and contrapositive should also be given some attention; pupils should be warned of the common errors of assuming the truth of the converse or inverse of a proposition which has been proved. Non-geometric illustrations can be brought in to enrich the unit. Some practice should be given in the indirect method of proof. In one of the schools conducting the experimental course, the approach to indirect proof through the law of the contrapositive proved quite successful.

Now, all this work takes time but it is time very well spent. It reaches to the very heart of the deductive process. Teachers who feel that they have not covered much ground during such lessons can satisfy their professional consciences by realizing that such activities (which may not require the use of paper, pencil or chalk) will yield more lasting benefits for our pupils in developing critical mindedness and a deeper appreciation of the power of mathematics. Unfortunately the present type of Regents examination hardly encourages these procedures.

It has often been pointed out that the traditional course in plane geometry deals little, if at all, with the 3 dimensional world in which the students live. The reason for that can perhaps be traced to an excessive preoccupation with minutiae of content and logical rigour to the detriment of the needs and interests of the adolescent boy and girl. Some work in building up the spatial imagination is started in the Junior High School course but it is not carried further in the tenth year. For pupils who cannot take a full course in solid geometry such work is especially important. There is no reason why we should not introduce the concepts and some laws of solid geometry concomitantly with the analagous concepts and laws of plane geometry. Thus the concepts of dihedral angle, parallel planes, perpendicular planes, a line perpendicular to a plane can be introduced in connection with parallel lines. Similarly theorems in solid geometry (e.g. the Pythagorean theorem extension, areas and volumes of prisms, pyramids, cylinder, cone and sphere) can be introduced informally, whenever the opportunity presents itself. Moreover many of the concepts of plane geometry find good exemplifications on solid figures. Thus isosceles and equilateral triangles, parallelograms, rectangles, circles etc. abound in the pyramid, prism, cylinder, cone and sphere. Problems in geography and simple navigation, which are becoming indispensable for an understanding of a world dominated by the airplane, can be clarified with the aid of the geometry of the sphere.

The treatment of solid geometry units should be informal and much use should be made of models (many of which can be constructed by pupils and they serve as real problems). Mensuration, as indicted previously should be applied in real settings. Locus ideas should be lifted out of the plane and brought into the third dimension. The following simple loci can be considered informally: Points at a given distance from a given fixed plane, points at a given distance from a given fixed line, points equally distant from 2 given fixed points or from 2 parallel planes or from 2 intersecting planes or from the circumference of a circle.

The New York State Education Department Bulletin of 1942 revealed: "Of all subjects in mathematics the deficiency

in arithmetic was most pronounced, and this was increasingly apparent in the upper six years of the secondary school and college. This is no doubt due in large measures to ineffective teaching in grades 1–6 and also to the lack of a planned program in later years to keep under review the skills acquired in these first six years." A national committee assigned to prepare war courses in mathematics advocated the inclusion of a great variety of practical problems employing the principles taught in these courses so as to furnish additional drill in arithmetic operations. The New York City H. S. Division also directed (in circular 45 of 1942): "Every teacher of mathematics is to assure himself of the competence of his pupils on the fundamental operations with whole numbers, common and decimal fractions." Recently the High School Division asked the Mathematics Chairmen's Association of New York City to study the problem of remedying the situation in the tenth year, during the first part of which, under the present course of study, so little opportunity is afforded to pupils in this much needed practice in arithmetic. In the experimental course, mentioned before, a considerable amount of arithmetic drill is furnished through the following. Many experiments require the pupils to perform computation with measured data. In this course the topic of similar triangles is advanced to the first semester thereby furnishing opportunities to use the algebra and arithmetic to solve problems involving proportions and square root. In doing this many of us feel that we are not thereby making geometry a handmaiden of arithmetic because the pupils is thus being equipped, in the first term, with a powerful technique of solving problems in indirect measurement. It also serves to compare and contrast congruence and similarity because of their proximity. This, we feel, will be at least a partial answer to the charge of New York State Education Department which said: "The processes of arithmetic and algebra have been almost entirely ignored in the courses in geometry. It is suggested therefore that more attention be given to numerical work in geometric problems involving these skills and that more emphasis be given to algebraic proof of geometric theorems and exercises." We have implemented these suggestions in the experimental course by including the Law of Sines (which has so many practical applications that the pupils had to solve theretofore by the less reliable scale drawing method).

The topic of locus can be enriched by the analytic description of several of the loci. Breslich has put it aptly as follows: "Next to trigonometry, analytic geometry is probably the most attractive course of elementary college mathematics. The use of the simple concepts of both would not only aid the pupil in the study of geometry, but would impress upon him the fact the most interesting mathematics courses are still ahead of him." (THE MATHEMATICS TEACHER, Nov. 1938, page 315.)

To reassure those who fall back only too often upon the vague cry of "preserving the basic or fundamental aspects" of our subject, it can be asserted with confidence that the new materials proposed and the new emphasis advocated will help preserve better than ever the real fundamentals of our subject. The word "basic" which seems to connote the inclusion of all the subject content that is traditionally taught should rather be employed to connote the main values of our subject at any particular time. However a far more important usage of the term should be in connection with what is basic to the child's development in relation to the times he is living in and the degree of his maturation.

SUMMARY

To summarize, this is what is advocated for the new age of Geometry for All American Youth:

1. We should exhibit geometry as an elementary illustration of the scientific

method by encouraging experimentation and clarifying the complementary roles of deduction and experiment.

2. We should exhibit the power of deduction by emphasizing its more significant applications, such as the explanation of indirect measurement, the study of the theory of simple instruments and practical applicances. Drill in the solution of the traditional 'riders' (which we call skeletal exercises because they are devoid of meat) should be deemphasized and be used largely to provide the necessary skill in applying the 3 or 4 general methods which arise in significant applications.

3. Far greater stress should be placed on the understanding of the concept of nature of proof with a corresponding deemphasis of the ability to reproduce proofs of theorems for its own sake.

4. We should lift the pupil from the second to at least the third dimension by showing him how frequently the applications of plane geometry principles have three dimensional settings and by discovering the three dimensional analogues of two dimensional theorems and loci.

5. We should provide practice on more useful constructions; we should encourage the use of the marked ruler, protractor, T-square, square edge; more problems should be included in scale drawing, vectors, designs and models (with their deductive explanations); we should liberate ourselves from the classical restrictions to straight edge and compasses and thereby obviate the constant need for apologizing to our pupils for what they may not do.

6. We should modify the sequence of topics (and this does not entail doing violence to the essential sequence to be found in most of the current textbooks) to permit a more gradual approach to formal deduction, to select the easier and more widely used topics in the first semester of the course, and to provide a greater amount of algebraic and arithmetic drill in the first semester.

This proposed enrichment of the subject through the juxtaposition of the manual and the mental, the pure and the applied, the concrete and the abstract, the inductive and the deductive, the synthetic and the analytical should leave a more permanent, a more valuable and a more pleasant impression upon most pupils than has been the case heretofore.

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The Geometric Form as an Architectural Matrix

By Charles R. Salit, *Floral Park, N. Y.*

Writers in architecture often muse over the origins of—what they prefer to call—certain architectural forms. These forms, according to a seeming consensus of opinion, came into existence by the grace and through the ingenuity of the local architect. He devised them, it is explained, in answer to some pressing demand to meet a specific need. Thus, in Auvergne, southeastern France, he invented cone-shaped roofs so as the better to meet the weather conditions peculiar to that region. In the case of the Persian tomb towers, of the eleventh and twelfth centuries A.D., the architect, with no such utilitarian purpose in mind, experienced the inexplainable urge to copy the supposedly conelike tent of the desert nomad. And similarly the Egyptian pyramid, according to architectural writers, were the product of the attempt and travail by the local architects to build for the pharoah a glorious tomb. Some of these writers have even gone out on the limb and described how the pyramid idea was inadvertently discovered by the evidently aimless process of superimposing receding mastabas, as it is supposedly exemplified in the step pyramids of Sakkara and Medum. And so the Mayan pyramids on the American continent had likewise a more or less casual birth, without antecedents or breeding.

The story of the column (cylinder) is very much the same as that of the pyramid and the cone. It is but another testimony of the inexplicable, almost unfathomable ability of man to create at will and at the drop of his proverbial hat, so to speak, forms that did not exist before. With such unlimited, creative powers at his instantaneous command, it indeed would be superfluous and unnecessary for the Greek architect to look around for older models, as for instance the plant columns of Egypt.

A similarly flippant treatment is accorded to the rest of the so-called "architectural" forms of this category. It is brought into being by a sort of sleight of hand of the architect under the not too genteel proddings of a utilitarian necessity, the much overrated and over publicised mother of all invention.

It seems that it had never occurred to the proponents of these views that the stock of this type of forms has long since been exhausted and that no new ones have been added within the memory of man. Were they but ordinary architectural forms that one can invent and create at will at the promptings of "mother necessity," no limits need exist. A careful investigation, however, will disclose that these forms, far from being ordinary, are extraordinary and unique and belong to a distinct class made up of the primary geometric forms. Now, the primary geometric form is not a mere architectural blueprint! It is all that and more! In fact, it is the basic blueprint or the mode of life by which man rose above an animal-like condition. In short, it is the matrix of civilization. Without it man could exist only as an animal, if at all! For even animals instinctively cultivate some of the geometric forms. Of people who have only a limited knowledge of the practical application of the geometric form we speak, as a rule, with contempt and say that they live like animals. The extent of application of the geometric form to some practical, useful end is the index of any one civilization at any one time.

Now that we have identified the architectural matrix as a primary geometric form, it is necessary to study some of its outstanding properties. First of all, it possesses certain qualities that are best described as static. A large group of these matrices, of the order REGULAR, has no power of variation other than that of size.

The other group of IRREGULAR forms has a double power of variation, namely that of size and shape. This latter property, however, in the case of the triangle, is fixed within narrow limits and, when applied indiscriminately to the rest of the forms of this class, may destroy their identity.

While the geometric forms themselves have a limited power of variation, their power of combination, on the other hand, is without limit. It is, therefore, only natural that the architect should be largely, if not entirely, concerned with the unlimited possibilities of combination. And architecture itself may be defined as THE ART OR PROCESS OF SELECTION, COMBINATION, AND DRESSING OF THE GEOMETRIC MATRIX. The geometric forms one selects, the order in which one combines them, and the way one dresses them, determine an architectural style. The architect no more invented the geometric form than he did the stones he built with.

Our inquiry into the nature of an architectural matrix now calls for a definition of the geometric form. This definition, however, we prefer to state in terms of the following theorem that requires a proof: "THE PRIMARY GEOMETRIC FORM IS A NATURE FORM THAT MAN BY DIVINE NECESSITY SINGLED OUT FOR RELIGIOUS WORSHIP." These forms differ from all other forms in nature in their static, almost changeless condition, and in certain qualities as well that move men to divine worship. Accordingly, at one time or other, every primary geometric form[1] in nature was a god-form and was worshiped as a god. Under certain conditions and at a certain cultural stage, man recreated by divine necessity his religious experiences. Some primitive people to this day still relive and reenact such experiences of a remote past. Remnants of this sort of practice are also discernible in the so-called higher civilizations, such as ours. In ecclesiastical and semiecclesiastical architectures the part played by the pyramid, the cone, the cylinder (column), and other primary forms, is too well known to require further elucidation.

That the geometric form possesses certain qualities that move men to divine worship is but another way of saying that man himself possesses these certain qualities that urge him to seek and worship his god. Out of this complementary, divine affinity, between man and certain forms in nature, evolved the geometric form. It evolved as a god-form pure and simple to satisfy a spiritual need. Among primitive people one still can find the geometric form at this stage of its spiritual evolution. To the American Indians the disk (circle) was a deity, or at least a symbol of the sun, and as such had not as yet learned how to do any physical work: it had to be carried by the worshiper, as some of our Indians still do in their sun ceremonies. Similarly in ancient Egypt the column (cylinder) was in this semi-divine state of repose as an Osirian symbol exacting religious adoration instead of doing some practical work as a machine. In the funerary shrines of the Sakkara step pyramid, the column (cylinder), far from supporting any weight, had to be held by a wall to which it was attached.

How fundamental this religious experience was, not only in the evolution of the geometric form but of man as well, is readily seen from even a superficial observation of animal architecture. I believe it is safe to presume that the animal lacks this sort of experience. Whatever instinctive knowledge of some of the geometric forms it may possess, this it acquired by force of material necessity in the struggle for self-preservation. Yet this sort of necessity has led only a few species to the partial discovery of some isolated geometric forms with whose range and power of variation, not to mention combination, the animal is totally unfamiliar. Furthermore, this necessity has restricted the employment and the use of the geometric form

[1] Such as the circle, the sphere, the cylinder, the cone, the pyramid, and a few others.

solely to the twin-purpose of self-preservation and self-propagation, a narrow, lopsided experience, bare of all cultural values. For the partial, and in some cases, the total lack of the geometric form, animal architecture is doomed to a static condition. Although the animal needs shelter and protection as much as does the primitive man, this sort of necessity, however, did not lead it to the discovery of the architectural matrix that alone could provide it with an adequate and progressive housing. The animal needs as much as the primitive man to cross a ravine in order to secure food, yet this sort of necessity did not lead it, with the possible and qualified exception of some insects, such as the spider, to the discovery of a bridge! Relatively speaking, religious necessity is more intense and persistent, and man feels it more deeply than were it merely utilitarian, and therefore it follows that the former is a more fruitful source of inventiveness than the latter. The underlying ideas of all basic inventions have invariably religious origins.

Although the geometric form is the basic landmark that differentiates man from animal, the dividing line is not, however, so apparent before the discovery that the god-form is also capable of physical work. The knowledge that coal exists is of no material value until it is discovered that it can be used as fuel! The difference between a savage and a civilized man chiefly lies in the degree of knowledge of the practical usefulness of the geometric form. Thus the American Indian, very likely, was familiar with all the primary geometric forms, but some of these he knew in a religious sense only. Although he was familiar with the circle as a sun fetish, he had not as yet discovered that it could be also put to work as a wheel![2]

[2] Charles R. Salit: "The Habitat of Geometric Forms," THE MATEHMATICS TEACHER for November, 1942. The Indian was, very likely, familiar with forms like the hemisphere (igloo), the cone (tipi), the cylinder (column), the prism (pillar), the pyramid (sun symbol), and the circle (sun disk).

The principles I have cited here are necessary and sufficient to the archaeologist and the architect to determine the degree of civilization of primitive and ancient peoples. These principles are simple and specific enough to serve as convenient guides in the examination of the monumental architectures of bygone days. We shall start out with certain premises that necessarily follow from the properties of the geometric form and certain well accepted procedures. In the first place, aside from stone implements, monumental architecture alone can leave behind it some tangible, archaeologically significant trace. This follows from the general rule that a civilization is judged by its highest possible achievements and not by that which is considered mediocre and vulgar. In the second place, monumental architecture can flourish only under the auspices of a political state. To construct a pyramid, like that of Khufu, required not only the vast resources of a powerful state but even more so the unquestioned divinity of a pharaoh!

These facts alone presuppose a certain kind and a certain degree of culture. In the third place, any people capable of monumental architecture were necessarily familiar with the geometric form as an architectural matrix. They had acquired this knowledge either from some other, more advanced, people, or in the course of their own nature worship, in which the primary geometric forms made up the paraphernalia of the ritual. In the fourth place, a differentiation between the architectural matrix as a geometric form and architecture as THE ART OR PROCESS OF SELECTION, COMBINATION, AND METHOD OF DRESSING MATRICES, will place the subject matter under consideration in its proper category. This classification proceeds from the principle that the geometric form has a unique existence, independent from architecture, and that the architectural matrix is but one functional aspect of the geometric form employed to this end. In the fifth and last instance,

each category must be studied and appraised in the light of its own antecedents. Without this differentiation it is difficult to tell whether the architect is talking about the origin of a certain kind of roof or the origin of the cone itself. The chances are that he himself is not any too sure as to what he really meant. Similarly we cannot talk in the same breath about the origin of the Persian tomb-towers and the origins of the cone and the cylinder, for the simple reason that they belong to two different categories of unique and distinct antecedents.

The subject of the great pyramids of Egypt offers us another illustration typical of the confusion that is bound to arise without a proper differentiation. Clearly, these pyramids are nothing but architectural matrices in whose construction engineering played a far bigger role than did architecture! Although the architect was called upon to select and fix the proper power[3] of variation of which the pyramid is, both culturally and physically, capable, and one that would also satisfy the megalomania of an ambitious pharoah, otherwise his task was largely confined to the design of the subterranean galleries and the construction of the funerary temples and chambers. Except for the Cyclopean size of the pyramid, the architect here followed throughout a design established by ancient and religious custom.

In the Mayan and Toltec pyramids, architecture is much more in evidence than it was the case in those of Gizeh. The surface construction is here more elaborate: in most instances, four grand staircases lead up to the sun temple perched on the truncated summit of the pyramid.

I doubt, however, that the American Indian originated a single geometric form! Accepting, as we necessarily must, that he came from Asia, across—what is now—the Straits of Bering, he brought along with him, as his cultural heritage, an elaborate paraphernalia that sun worship had evolved in the far reaches of the Himalayas over a period measured in tens of thousands of years. In his ritualistic repertoire, the pyramid represented the Sun God ruling as a king over a political state. Not until the Indian of Central American had reached the stage of a political state ruled by a sun-king, could the pyramid appear as a matrix of the monumental architectures found at Chichen Itza and Tikal. Although there are some Oriental vestiges present in the Mayan pyramid architecture, on the whole and by far it is indigenous and the product of the American Indian.

We have identified the architectural matrix as a geometric form: the converse, however, does not adequately define the geometric form! It is all that and more! The cylinder, the circle (wheel), and the sphere (ball-bearing) are the matrices of the machines of steel. Without these our high-geared machine age could not be thought of. In Christian theologies the concepts of a triune God (Father, Son, and the Holy Ghost) and of man as a physical, mental, and spiritual being, fail to produce the necessary conviction and perfect imagery without the aid of the equilateral triangle. Similarly the mark of godhood is the circle shown as a halo about the sainted head. The double triangle of the Hebrews, known as the "Shield of David" or as the "Seal of Solomon" has the recondite allusion to the Tetragrammaton or four-lettered, name of God.[4]

We could cite example after example, in a demonstrative fashion, how the primary geometric form,[5] far from being a mere architectural expression, is the fundamental matrix of all the vital activities that identify man and distinguish him from animal.

[3] The double power of variation in shape and size.

[4] Charles R. Salit: "The Habitat of Geometric Forms," THE MATHEMATICS TEACHER for November, 1942, New York.

[5] Those forms in nature that man by divine necessity singled out for religious worship. Probably nine in all.

Ptolemy's Theorem and Regular Polygons

By L. S. SHIVELY, *Ball State Teachers College, Muncie, Ind.*

1. INTRODUCTION. It might be both profitable and interesting to attempt to evaluate the relative importance of some of the outstanding theorems of elementary geometry by means of a poll of mathematicians. If this were done, there is little doubt that the theorem of Pythagoras would in the opinion of most people, head the list. It is not so clear just how other important theorems would be ranked. But without doubt, Ptolemy's theorem concerning an inscribed quadrilateral is deserving of a high place. This judgemnt is warranted, both from the point of view of the use which Ptolemy made of it, and from a consideration of its power as a tool for geometric study.

Ptolemy's theorem is fundamental in the mathematical treatment of the geocentric theory of the universe as it was developed about a century after the beginning of the Christian Era. This theory, although now in the discard, stimulated scientific work and influenced thought for 1400 years.

In what follows, the theorem itself will be considered, the use which Ptolemy made of it will be explained, and application of it to the study of the regular polygons will be made.

2. PTOLEMY'S THEOREM. *If a quadrilateral is inscribed in a circle, the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.*

Two proofs will be given. The first one is based upon similarity, the second on the law of cosines.

First proof. Referring to Fig. 1, let BE be drawn so that $\angle ABE = \angle DBC$. Then the triangles ABE and DBC are similar, from which it follows that

$$AE = \frac{AB \cdot CD}{BD}.$$

Also, triangles BCE and ABD are similar; and

$$EC = \frac{AD \cdot BC}{BD}.$$

Adding these equations and clearing fractions gives the desired result:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

Second proof. Designating the sides and the diagonals of the quadrilateral by the

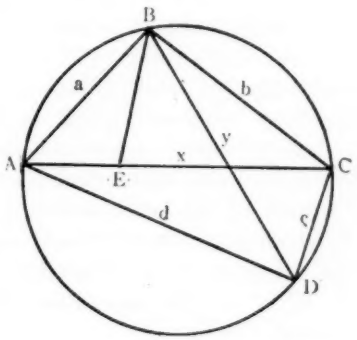

FIG. 1

lower case letters as indicated in the figure and applying the law of cosines to triangle ABC gives

$$x^2 = a^2 + b^2 - 2ab \cos ABC.$$

Also, from triangle ACD

$$x^2 = c^2 + d^2 - 2cd \cos CDA.$$

The last equation is equivalent to

$$x^2 = c^2 + d^2 + 2cd \cos ABC,$$

since angles ABC and CDA are supplementary. If now we multiply the first and third equations by cd and ab respectively and add, we get

$$(ab + cd)x^2 = (a^2 + b^2)cd + (c^2 + d^2)ab.$$

Dividing and factoring,

$$x^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

In a similar way, with triangles ABD and BCD,

$$y^2=\frac{(ac+bd)(ab+cd)}{ad+bc}.$$

Multiplying and extracting square roots, gives

$$xy=ac+bd.$$

We note that in the foregoing development are equations which enable us to give the lengths of the diagonals in terms of the sides.

3. Ptolemy's Use of the Theorem. Although the first complete text on trigonometry did not appear until 1533,[1] its principal concepts and theorems were developed by various writers at much earlier dates. Ptolemy organized and arranged much of the work which had been accomplished by his predecessors, in the form in which it is found in the Almagest.[2] This was done in order that he might avail himself of it in the exposition of his astronomical theory.

One of Ptolemy's needs was a table giving the lengths of chords of a circle corresponding to various central angles. A little thought will show that this is the equivalent of our present day table of sines. In order to construct this table he used the relation which in modern terms is known as the addition theorem for sines: $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

To show how this was accomplished we first observe (Fig. 2) that if an angle is inscribed in a circle whose diameter is unity, the sine of the angle is represented by the chord which subtends the angle. Thus, $\sin\theta=AC$; for, if AB' is a diameter,

$$\angle AB'C=\theta, \quad \text{and} \quad \sin\theta=\frac{AC}{AB'}=AC.$$

Now in Fig. 3 we take an inscribed

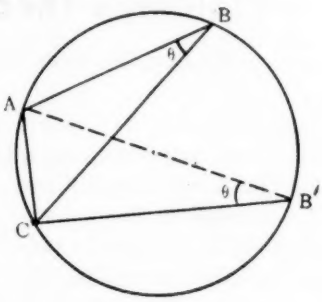

Fig. 2

polygon, one of whose diagonals is a diameter, and for the sake of simplicity, the unit of length is chosen equal to the diameter of the circle. By Ptolemy's theorem,

$$xy=ac+bd.$$

Now $x=1$, $y=\sin(\alpha+\beta)$, $a=\sin\alpha$, $b=\cos\alpha$, $d=\sin\beta$, $c=\cos\beta$. Substitution of these values gives the result:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

With the aid of this, expressed in terms of the lines of the quadrilateral, rather than sines and cosines, Ptolemy computed a table of lengths of chords.

4. Application to the Rectangle. If the inscribed quadrilateral is a rectangle whose adjacent sides are a and b and whose diagonals are c, the theorem reduces to the Pythagorean theorem: $a^2+b^2=c^2$. Since the first proof given in Sec. 2 is entirely independent of the theorem of Pythagoras, we have here a simple proof of the latter.

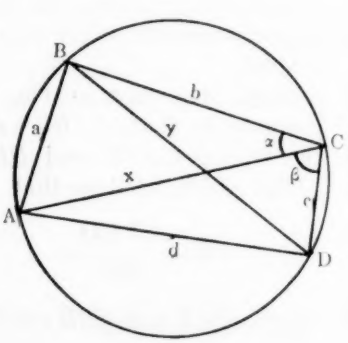

Fig. 3

[1] This was the book by Johann Müller, who also went by the name Regiomontanus, entitled *De Triangulis Planis et Sphericus cum Tabulis Sinuum.*

[2] See M. Halma, *Composition mathématique de Claude Ptolémée*, vol. 1, pp. 26–46.

5. The Regular Pentagon. Consider a regular pentagon inscribed in a circle whose radius is r. Let a be the length of a side and d the length of a diagonal of the

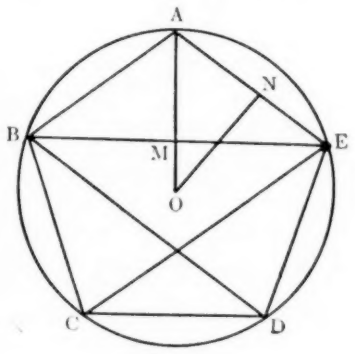

Fig. 4

pentagon (Fig. 4). Applying the theorem of Ptolemy to the polygon $BCDE$, we have

$$d^2 = ad + a^2.$$

The solution of this quadratic equation for d in terms of a gives $d = \frac{a(1+\sqrt{5})}{2}$.

Next, draw the perpendicular AM to the diagonal BE. In the right triangle ABM we have, from the foregoing and from the Pythagorean theorem: $AB = a$,

$$BM = \frac{a(1+\sqrt{5})}{4}, \qquad MA = \frac{a\sqrt{10-2\sqrt{5}}}{4}.$$

And since $\angle MBA = 36°$, $\sin 36° = \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}$; $\cos 36° = \sin 54° = \frac{1+\sqrt{5}}{4}$.

To find the relation between the side of the pentagon and its radius, consider the right triangle OAN. Since the angle $NOA = 36°$, $AN = r \sin 36°$, and the side

$$a = \frac{2r\sqrt{10-2\sqrt{5}}}{4} = \frac{r\sqrt{10-2\sqrt{5}}}{2}.$$

The radius of the inscribed circle is

$$ON = r \cos 36° = \frac{r(1+\sqrt{5})}{4}.$$

The area of the pentagon is

$$5 \cdot ON \cdot AN = \frac{5r^2\sqrt{10+2\sqrt{5}}}{8}.$$

In terms of the side a, the area is

$$\frac{a^2\sqrt{25+10\sqrt{5}}}{4}.$$

6. The Regular Hexagon. If the vertices of a regular hexagon whose side is a be numbered 1, 2, $\cdots$ 6, and the theorem is applied to the quadrilateral 1346, we obtain the equation $x^2 + a^2 = 4a^2$, where x is the length of a short diagonal of the hexagon. Thus $x = a\sqrt{3}$.

7. The Regular Heptagon. For simplicity, let the side of the heptagon have the length unity, let the vertices be numbered consecutively from 1 to 7, and represent the diagonals 1 3 and 1 4 by x and y respectively. Then from quadrilaterals 1 2 3 7 and 1 3 4 7 we obtain the equations

$$x^2 = y + 1,$$
$$y^2 = xy + 1.$$

By elimination of y from this pair it is found that x must satisfy the cubic equation $x^3 - x^2 - 2x + 1 = 0$.

Inspection shows that this equation has no integral root. Hence it has no rational root. It follows that x cannot be constructed with the ruler and compass,[3] and that, consequently it is impossible with ruler and compass to construct a regular polygon of seven sides.

8. The Regular Polygon of Nine Sides. Designating the vertices of the polygon whose side has unit length by the numbers from 1 to 9, and the diagonals 1 3, 1 4, 1 5 by x, y, z, respectively, we obtain in succession from the polygons 1234, 1245 and 1456 the equations

$$x^2 = y + 1,$$
$$y^2 = xz + 1,$$
$$xz = y + z.$$

[3] See Dickson, *Theory of Equations*, pp 90-92, Wiley, 1914.

From these equations, by elimination of y and z, x is found to satisfy the equation $x^3-3x-1=0$. By reasoning similar to that at the end of Sec. 7 it follows that a regular polygon of nine sides cannot be constructed by means of the ruler and compass.

From the first and third of the above equations it is seen that $z=x+1$. This gives the interesting result: *A longest diagonal of a regular nonagon is equal to the sum of a shortest diagonal and a side.*

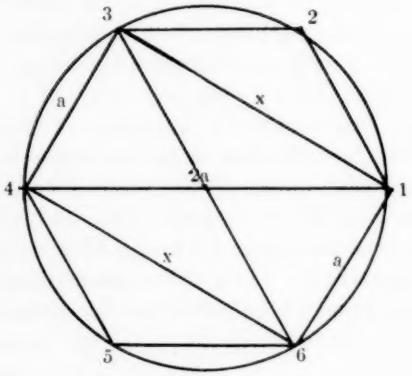

FIG. 5

9. THE REGULAR POLYGON OF SEVENTEEN SIDES. This polygon has diagonals of seven different lengths. As before, the length of a side is taken as the unit and x, y, z, t, u, v, w, represent the diagonals in order of increasing length. The polygons 1234, 1345, 1456, 1567, 1678, 1789, 189 10 lead to the equations

$$x^2=1+y, \qquad xy=x+z,$$
$$xz=y+t, \qquad xt=z+u,$$
$$xu=t+v, \qquad xv=u+w,$$
$$xw=v+w.$$

From the first six of these we obtain the following values of $y, \cdots, w$, in terms of x:

$$y=x^2-1$$
$$z=x(x^2-2),$$
$$t=x^4-3x^2+1,$$
$$u=x(x^2-1)(x^2-3),$$
$$v=x^6-5x^4+6x^2-1,$$
$$w=x(x^2-2)(x^4-4x^2+2).$$

If the last two of these are substituted in the equation $xw=v+w$, the result is $x^8-x^7-7x^6+6x^5+15x^4-10x^3-10x^2+4x+1=0$, one root of which is obviously

$$2\cdot\cos\frac{\pi}{17}.$$

Knowing the Gauss theory of the 17-sided polygon, it is clear that the shortest diagonal can be constructed with ruler and compass. But there appears to be no conveniently deduced method of making the construction, based upon the foregoing equation of the eighth degree.

The work shows that each diagonal of the polygon is a rational integral function of the shortest diagonal.

10. FURTHER RESULTS. Some of the results just given generalize for the regular n-gon. Thus, for example, it is easily seen that all the diagonals of a regular n-gon are expressible as polynomials of the shortest diagonal.

By properly interpreting the equations $x^2=1+y$, $xy=x+z$, $xz=y+t$, $\cdots$ of Sec. 9, the variables may be regarded as follows: $x=2\cos\theta$, $y=1+2\cos 2\theta$, $z=x+2\cos 3\theta$, $\cdots$. Making the necessary eliminations, there result the multiple angle formulas for the cosine,

$$\cos 2\theta=2\cos^2\theta-1,$$
$$\cos 3\theta=4\cos^3\theta-3\cos\theta, \cdots.$$

Attention, National Council Members

BE SURE to mark your ballot (page 92, February 1946 issue of THE MATHEMATICS TEACHER) and send it to Secretary E. W. Schreiber, State Teachers College, Macomb, Ill. before April 15, 1946.

A Suggested Program for Teaching the Function Concepts in High School Algebra*

By W. G. Hendershot, *Roosevelt High School, Chicago, Ill.*

Need for Better Instruction

We are all familiar with the complaint of teachers of other subjects, especially physics, that their pupils do not know mathematics, not even the simplest kind of manipulation involving nothing more than the most elementary forms of arithmetic and algebra. If such criticism is unjustified, we should defend ourselves and produce facts to support the competence of our instruction. Often we have been too complacent about this. We have been too content to go serenely along our way assuming a lofty, unconcerned air as if to say, "Are we not experts in the field of mathematics?" or "Isn't it a little presumptuous on the part of teachers in other fields to criticise us?"

Thus, since we have not always been too concerned with the effectiveness of instruction in mathematics, the result has been that mathematics in the high schools was in bad repute. This was true especially in the years preceding the war. The general educator was rather contemptuous of the whole mathematics program. An examination of books on secondary education shows that the general attitude may be stated something like this: "Mathematics has had a long and fair trial and the results are disappointing." Moreover, we have thought this outside criticism rather strange and overdrawn, for we have had the same pupils of whom the physics teacher complained in mathematics classes, and we were quite sure they did know their algebra and geometry.

Now I was brought into close contact with this whole problem and thus was able to view it from both sides. For quite a number of years I worked in a small school where we had one physics class which I taught in addition to mathematics classes. It happened many times over the years that I had the same pupils in both mathematics and physics. As a physics teacher I was able to view my work as a mathematics teacher, and the results were poor. As a physics teacher I concluded there was something wrong with me as a mathematics teacher. Either that or my approach to the teaching of algebra, together with the type and arrangement of subject matter as found in the conventional text, was wrong. At least I was conscious of a problem, for the pupils' knowledge of algebra simply didn't carry over into allied fields. At this point let me express the opinion that the best thing that can happen to the conscientious teacher of algebra is to have, occasionally, a class in physics. There he can see the results of his teaching when the pupils' knowledge must be applied to real problems.

There are some who say that physics need not be mathematical in its subject matter. I do not hold to that view, for we are constantly using the language of mathematics which requires quantitative thinking. Such questions as how much? how far? how fast? how heavy? how many? are always being raised. Also we have variables, constants, relationship, dependence, limits, ratio and proportion, and the formula. These are good mathematical terms and the average pupil has insufficient equipment for dealing with them.

As examples of these terms—

Variable: Pressure varies as the depth.

Pitch of a note varies with the frequency of vibration.

* Presented to the Men's Mathematics Club of Chicago and the Metropolitan Area on February 16, 1945.

Rate of evaporation varies with temperature, amount of surface exposed, condition of the air (relative humidity), kind of liquid.

Constant: Tables of densities.

Tensil strength.

Heat of liquids—5 constants; boiling point, freezing point, heat of fusion, heat of vaporization, and specific heat.

Limits: Hook's law applies only within the limits of perfect elasticity.

Absolute zero.

Saturation.

Law of mixtures—Heat transference measurable only within certain limits.

Ratio, proportion, and per cent:

direct linear

$$P=hd, \quad W=Fd, \quad I=\frac{E}{R}, \quad V=gt$$

quadratic

$$S=\tfrac{1}{2}gt^2, \quad \text{C.F.}=\frac{mv^2}{gr}, \quad \text{K.E.}=\tfrac{1}{2}mv^2$$

inverse

$PV=K$, 1st Class Lever

inverse quadratic, law of gravitation; magnetic force.

Per cent: efficiency, relative humidity, and several other relations expressed as a per cent.

These are all formulas found in any high school physics text.

You may say that nearly all of the relations and situations in physics are expressed in formulas so that all you have to do is to evaluate a formula as in algebra. But it isn't as easy as that. True, if given

$$I=\frac{E}{R},$$ and $E=110$, $I=.5$, find R, most

pupils can do so just as in algebra; but if the situation is involved in a verbal statement, it is harder. Or again, if you ask a pupil to use an ammeter to find the resistance of a lamp on a standard 110V. circuit, he will have trouble. There are two difficulties, he can't translate the verbal statement into a formula nor identify the various elements in the formula.

As an example take the Boyle's Law Experiment. Here we have the four-fold tieup of the algebraic process applied to the experimental method.

1. The collection of data arranged in a table in the form of number pairs. For each pressure there is a corresponding volume.
2. The graphing of the number pairs.
3. The statement of the law. Verbal statement.
4. The law expressed as a formula. $VP=K$. Here we have the relation between a body of gas and the corresponding pressure expressed in four different ways. The students' experiences in high school algebra is of little use, because the various methods of expressing relationships have not been correlated.

What's Wrong with the old Type of Organization of Algebra?

In the old type of organization we have little more than blind manipulation of algebraic processes and symbols. This is without sense, reason, or meaning, and the results are without significance.

To substantiate the above statement let me submit a few observations.

1. We all have experienced the difficulty of getting the pupils to check their results. They simply will not check their answers. Why? Because to them correct answers are not so very important as the problem is concerned only with abstract numbers. They have little curiosity as to the correct answer and receive little satisfaction upon verifying the correctness of their results.

2. Upon starting a set of problems how often have we heard this remark, "Just show me how to do the first one, and then I can do them." That means no thinking is required, just blind manipulation of symbols. Neither do the various problems develop mathematical situations requiring analysis or interpretation.

3. Verbal problems which require some thinking are to our modern pupils definitely "out." It has been my experience that a beginning class in algebra abso-

lutely refuses even to try word problems. They go on a sit-down strike. Here are problems that require thinking, and the pupils are not prepared to think. In fact when it comes to thinking, they are conscientious objectors. They don't believe in it. But the question is, who made them that way?

4. Signed numbers. We don't put meaning into signed numbers but plunge into manipulating them according to rule. So what happens? It has been my experience that after a few days the pupils can add signed numbers quite well. Then we go into substraction and after a few days they can neither add nor subtract.

5. We fail to get good graphs and to use graphs as a means of giving information. These two factors are related, for if we fail to use graphs to impart information why construct them accurately? Some problems can be solved only by the graphical method in elementary mathematics, as finding the resultant of two forces acting at an angle. Graphs are not a subject of study in one chapter only but should be used throughout the entire course. Note that practically every one of the suggested problems which follow may be graphed.

What to Do About It? Organization of the Course

Since the function concept is the most fundamental thing in science and for several hundred years has dominated advanced mathematics, it would seem that it should be brought to the high school level. It should be used as an organizing principle so that pupils will have their algebra made meaningful through the recognition, and representation of functional relations between variable quantities.

Examples of function relationship are many, and many problems may be selected to bring out the idea of relationship, variation, and dependence, as is shown by the suggested list of problems submitted at the end of this article.

In such a proposed course the pupils will have to read and understand the language of mathematics. A great deal more reading matter will be presented. Verbal problems will not be an ioslated section of the book. The course may well open with practice in translating statements in ordinary language expressing quantitative relations into the compact shorthand of algebra. This leads the pupil in a natural way to the concept of general number. These statements lead to simple formulas of the linear type $y=ax$. Then, after sufficient practice in this work we pass on to the evaluation of letters in these formulas. This work in evaluation leads to the solution of equations, but at the same time by varying the values given one letter, we call attention to the variation in the values of the other. The matter of relationship between variables is allowed to develop gradually.

A large amount of concrete material can be used in the construction and evaluation of formulas of this simple type, and also for calling attention to relationships. From mechanics we have the relation between forces on an inclined plane, between velocity and the time of a falling body, between the extension of a spiral spring and the stretching force, and the distance between a body moving uniformly and the time. From business we have the relation between simple interest and the time and the various percentages relations. The relations may also be studied by means of tables and especially by graphs. (See problems 1, 2, and 3 at the end of this article.)

By adding the constant b to ax we get the linear relation $y=ax+b$. There are many examples of this type such as:

$$V=V_1+32t, \quad L=L_1+aw, \quad A=P+Prt.$$

These relations are studied not merely by evaluation of the letters of the formulas but by the tables of corresponding values and by graphs. This leads to such questions as the following. What effect does an increase of any number of times in the independent variable have on the depend-

ent variable? How much does the dependent variable change per unit of change in the independent variable? This is the slope of the line. These relations may be made concrete and meaningful by questions as in the examples. Here we see the significance of negative numbers and also the limits within which a formula is valid. (See problems 4 and 5.)

A problem involving two situations leading to a linear pair of equations may also be treated from the functional point of view.

The quadratic relation $y=ax^2$ occurs frequently in science, as previously noted in quadratic variation, and in geometry in the area formulas. Such questions as the following should be raised frequently. How is the size of a water pipe affected by doubling the diameter? Effect on area of square if side is doubled? tripled? Distance a body will fall if time is doubled? (See examples 6 and 7.)

Also $y=ax^2+bx+c$. The study of the general quadratic is approached by first adding a constant C to $y=ax^2$ forming $y=ax^2+C$ and then adding a constant to x forming $y=a(x+h)^2$. The tabular arrangement of number pairs and graphs are used to introduce these new constants. Examples of the first is any parabolic arch, and also $h=H-16t^2$. Ring or annulus area $A=\pi R^2-\pi r^2$. (See problems 8, 9, 10, and 11.)

The meaning and significance of the roots of a quadratic equation, whether they are real, equal, or imaginary, are made clear by such examples.

In the inverse relation $xy=c$ we have many examples from science, as:

$rt=d$, $lw=A$, $fd=w$, $hd=p$, $at=v$, $nl=v$

d, A, w, p, and v are constants.

Here we study the effect on one variable by changing the value of the other. (See example 12.)

This leads also to the inverse proportion as in the lever problem in physics.

The only objections that I have heard of such a proposed course as outlined is that it is too hard. To dip down into the middle of such a course is rather difficult, but if the pupil has been brought up on a functional diet from the beginning, it is not too hard. Moreover, whatever may be the ills of our present mathematical setup, I don't believe the solution lies in trying to make it easy. The truth is that for many people mathematics is difficult. We should rather try to make algebra meaningful, useful, and interesting. As far as my experience goes I have no doubts of the pupils' ability to handle this type of algebra profitably.

TYPE PROBLEMS

1. Sound travels at a speed of 1130 ft. per second. Use this fact to make a table of corresponding values of time t (number of seconds) and distance d (number of feet). Give t the values $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, 4, 6, 10, 20.
   (a) What is the value of the ratio d/t?
   (b) What is the formula corresponding to the table?
   (c) How many miles away is a stroke of lightning if it is 15 sec. till the sound of the thunder is heard?
2. The formula $p=4s$, expresses the fact that the perimeter of a square is a function of a side.
   (a) Make a table of pairs of values of the variable s and the function p in which the values of the variable have a common difference 2 and run from 0 to 10. If the values of the function have a common difference, find it.
   (b) Make a table starting with s equal to 1 in which any value of the variable is double the preceding one. How is each value of the function related to the preceding one.
   (c) If any value of s is multiplied by 3, what relation do you find between the corresponding values of p?
   (d) If a side s of a square is doubled, what effect does that have in the perimeter?

(e) If a side of a square is made n times longer, what effect does that have in the perimeter?

3. A north-south railroad passes through town T. The location of a certain train on this railroad is given by the formula $d=-40t$. Distance north, velocity north, and time in the future are positive quantities.
   (a) Find the time when the train is in turn 20, 60, 100 mi. north of T.
   (b) How is t affected when d is multiplied by 3, by 5?
   (c) Find the time when the train is in turn 20, 60, 100 mi. south of T.
   (d) How is d affected when t is multiplied by 3, by 5?
4. The formula $v=40-8t$ gives the velocity of a train t seconds from the instant the brakes are applied.
   (a) What is the velocity at the instant the breaks are applied?
   (b) Find the values of v for $t=1, 3/2, 2, 5/2, 3, 4, 5$.
   (c) What is the change in v per half unit change in t?
   (d) What is the rate of change of v?
   (e) What value has v for $t=6$? Is this value valid?
   (f) What is the range of values for t? for v?
5. If a stone is thrown upward with a velocity of 100 ft. per second, after t seconds its velocity v is given by the formula $v=100-32t$.
   (a) Find the velocity at the end of 1, 2, 3, sec.
   (b) Find the velocity at the end of the fourth second. How do you account for the negative value of v?
   (c) Find the velocity at the end of $6\frac{1}{4}$ sec. How does its *absolute* value compare with that of the initial velocity?
   (d) Make a table of corresponding values of t and v for $t=1, 2, 3, 4, 5, 6$.
   (e) What is the change in v per second? How do you account for the sign of your result?
6. The following table was obtained in testing the brakes of a car, the variables are v (miles per hour at the instant the brakes were applied), and s (number of feet required to bring the car to a stop).

If	$v=$	5	10	15	20	30	50
Then	$s=$	1.5	6	13.5	24	54	150

   (a) Complete the table and write the formula.
   (b) Use the table to find the number of times the distance is multiplied if the velocity is doubled; trebled; quadrupled.
   (c) If the driver of this car traveling at the rate of 40 mi. per hour should suddenly discover a stationary object on the road ahead at a distance of 120 ft. ,would he be able to avoid a collison?
7. A bomb is dropped from a height of 1600 ft. In how many seconds will it strike the ground? (use $S=16t^2$ and solve for t).
8. An anti-aircraft gun fires a projectile with an initial velocity of 800 ft. per second. When will the projectile reach the elevation of an airplane 10,000 ft. high? Will the projectile rise higher than the airplane? (use $S=vt-16t^2$).
9. An anti-aircraft gun projects a shell with an initial velocity of 1200 ft. per second at an airplane 8,000 ft. high. At what time after firing will the projectile be at the height of the airplane? Interpret the two values of t.
10. An anti-aircraft gun projects a shell with an initial velocity of 604 ft. per second at an airplane 8,000 ft. high. At what time after firing will the shell be at the elevation of the airplane? Interpret the imaginary values of t.
11. A projectile is fired vertically upward from a gun with an initial velocity of 1600 ft. per second.
    (a) In how many seconds will it reach its maximum height?

(b) What will be its maximum height?
(c) What will be its height after 10 sec.?
(d) When will it be again at this height?
(e) What will be its velocity after 80 sec.?
(f) What will be its height after 80 sec.?
(g) When will it strike the ground?

12. The area of a rectangle is 288 sq. in.
(a) What is the value of K?
(b) Construct a table of corresponding values of w and l for $w=6$, 12, 18, 24, 30, 36, 42, 48.
(c) Use the table to find the effect on the length if starting with $w=6$ we make w 2 times, 3 times, 4 times as large, and so on.
(d) Find the dimensions if the length is 8 times the width.
(e) Find the dimensions if the length is equal to the width.
(f) Find the dimensions if the length is 12 inches greater than the width.

Regional Meeting of the National Council of Teachers of Mathematics in Connection with the Meeting of the A.A.A.S., St. Louis, Mo. Saturday, March 30, 1946

Meetings will be held at the Melbourne Hotel

10:30 A.M.

Presiding—F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tennessee.

1. Whither Mathematics and Science. G. H. Jamison, Northeast Missouri State Teachers College, Kirksville, Mo.
2. More Desirable Residuals in the Geometry of Space. Miles C. Hartley, University High School, Urbana, Illinois.
3. Multi-Sensory Aids for Post-War Mathematics Teaching. E. H. C. Hildebrandt, Northwestern University, Evanston, Ill.
4. Discussion.

2:30 P.M.

Panel Discussion—Projects of the Commission on Post-war Plans
Presiding—Raleigh Schorling, University of Michigan, Ann Arbor, Mich.
Members of the Commission
Raleigh Schorling (Chairman)
William Betz, Specialist in Mathematics in the Schools of Rochester, N. Y.
William A. Brownell, Duke University, Durham, N. C.
Walter H. Carnahan, Purdue University, Lafayette, Indiana
Eugenie C. Hausle, Chairman of Standing Committee on Mathematics in New York City, James Monroe High School, Bronx 59, N. Y.
Virgil S. Mallory, Professor of Mathematics, State Teachers College, Montclair, N. J.
C. V. Newsom, Oberlin College, Oberlin, Ohio
Mary Potter, Supervisor of Mathematics, Racine, Wisconsin
H. Vernon Price, University High School, Iowa City, Iowa
William L. Schaaf, Assistant Professor of Education, Brooklyn College, Brooklyn, N. Y.
Rolland R. Smith, Coordinator of Mathematics, Public Schools, Springfield, Mass.
Ruth Sumner, Mrs., President, Mathematics Section, State Teachers Association, Oakland, Calif.
F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tenn.
James H. Zant, Professor and Acting Head, Department of Mathematics, Oklahoma A. and M. College, Stillwater, Okla.

Many Yearbooks Will Soon Not Be Available

BUY the National Council Yearbooks that you lack and need now. The first, second, and tenth are now out of print and most of the others will soon be gone. See page 144 of this issue for the complete list.—Editor.

Another Way to Teach the Work Problem

By WALTER BERNARD
Gilbert Stuart Junior High School, Providence 6, R. I.

IN THE April, 1945 issue of THE MATHEMATICS TEACHER William R. Lueck expounds with clearness and in detail a method of teaching the so-called "work problem." The solution taught is the same as that in most textbooks, when an example of a solution is set forth, but Lueck has shown teachers how to break down the attack into simple steps so that even slower-than-average pupils can learn how to solve work problems.

I have had some success teaching the solution of work problems from a different point of view. Since the work problem is a rate problem, it can be solved by the same method as that used in the solution of a motion, or "rate-time-distance," problem. In fact, work problems are easier to solve than motion problems, because they correspond to motion problems in which the distances traversed are always distances of one mile only. Once the pupil learns to think in terms of "jobs," "lawns," "tanks" etc. instead of only "one job," "one lawn," "one tank," etc., the similarity between work problems and motion problems becomes obvious.

Here are repeated the four work problems considered by Lueck,[1] each problem followed by a corresponding motion problem. This has been done because, if textbooks are criteria, the correspondence of work problems to motion problems is not generally appreciated. However, as far as the pupils are concerned, the work problem could be among their first examples of the rate concept. I have taught the solution of work problems like Problem No. 1 to eighth grade pupils who, of course, had no knowledge of relative motion problems.

1. *John can mow a lawn in two hours. Sam, who is younger and uses a smaller mower, requires five hours to mow the same lawn. In how many hours will the boys finish the lawn if they work together?*

1m. *A boat can travel one mile in two hours on still water. How long will it take the boat to go down stream one mile, if the stream flows one mile in five hours?*

2. *A tank can be filled by one pipe in four hours, and emptied by another in five hours. If both pipes are opened, how long will it take to fill the tank?*

2m. *A boat can travel one mile in four hours on still water. How long will it take the boat to travel one mile upstream if the stream flows one mile in five hours?*

3. *Mr. Clark can plow a field in twelve days when working alone. When assisted by his neighbor, the two together can plow the field in eight days. How long will it take the neighbor when working alone?*

3m. (Let's get on the slow boat again.) *A boat can travel one mile in twelve hours on still water. If the boat travels one mile downstream in eight hours, how long will it take the stream to flow one mile?*

4. *James can build a play house in eight days. After working three days he is joined by Henry, and the two together complete the work in two days. How long will it take Henry alone to do the work?*

4m. (Here we go for our last ride on the slow boat.) *A boat can travel one mile in eight hours on still water. After traveling for three hours, the boat enters a stream and travels down the stream for two hours. If the total distance traveled by the boat in both still water and current is one mile, how long does the current take to flow one mile?*

I suppose I am not an up-to-date teacher using slow boats when I could be riding in airplanes with head winds and tail winds, but I hesitated to do so with such slow rates. Even for boats the time is slow. However, boats or airplanes, mathematical continuity comes to our rescue, permit-

[1] Lueck, William R., "How to Teach the Work Problem," THE MATHEMATICS TEACHER, April, 1945, pp. 181–184.

ting us to use vehicles in many situations, impossible as well as possible.

Problem No. 1 is easy enough to be done by arithmetic, but since algebra is being taught, let us solve the problem by algebra. First the class is advised to think of not one lawn but of several lawns, all alike. Since we are doing the problem by algebra, we let a letter, h, stand for what we are trying to find, the hours it takes the boys together to mow one lawn.

The class is then asked, "At what rate does John mow lawns?"

With the first work problem this does not usually bring a response, so the question is reworded, "How many lawns an hour does John mow?"

This question seldom fails to arouse the pupils to a show of hands and the answer, $\frac{1}{2}$ of a lawn an hour. In the same way the pupils decide that Sam's rate is $\frac{1}{3}$ of a lawn an hour.

The pupils are next asked how many lawns are to be mowed and how long will it take. They do not seem to have any difficulty in giving the responses, 1 lawn and h hours respectively.

It is now possible for us to set up the equation, for the rate at which the lawns (1 lawn) are cut can be expressed in two mathematically independent ways: $\frac{1}{2}+\frac{1}{3}$ and $\frac{1}{h}$.

If this problem had been done by arithmetic, we would have found the time by dividing the number of lawns to be mowed 1, by the total lawns mowed an hour, $\frac{1}{2}+\frac{1}{3}$.

Problem No. 2 is reasoned in the same way as Problem No. 1, the emptying pipe filling the tank at $-\frac{1}{5}$ tanks an hour.

Problem No. 4 is the most involved of the four problems, but it should yield easily to reasoning by rates. We find James' rate of building play houses by dividing the number of play houses he builds, 1 play house, by the number of days it takes him to build them, 8 days. His rate is, therefore, $\frac{1}{8}$ play houses a day. Since Henry can build 1 play house in H days, his rate is $\frac{1}{H}$ play houses a day. Working 3 days alone, James will build $3\times\frac{1}{8}=\frac{3}{8}$ play houses, which subtracted from the total number of play houses to be built, 1, leaves $\frac{5}{8}$ play houses remaining to be built. Another way of expressing the number of play houses remaining to be built is as the product of the time remaining, 2 days, and the number of play houses built a day, $\frac{1}{8}+\frac{1}{H}$. These two mathematically independent ways of expressing the number of play houses remaining to be built are used to form the following equation expressed in the original amounts of the problem:

$$1-(3)\ \frac{1}{8}=2\left(\frac{1}{8}+\frac{1}{H}\right).$$

Problem No. 4 is an illustration of the fact that we can often set up the equation of a problem from several viewpoints. The rate at which "the play houses remaining to be built" may be expressed in two mathematically independent ways: $\frac{1}{8}+\frac{1}{H}=\frac{1-3(\frac{1}{8})}{2}$. Or, we could so express the total number of play houses, 1 play house, to be built:

$$(3+2)\frac{1}{8}+(2)\frac{1}{H}=1.$$

Of course, the fact that we are not limited to an expression of the unknown in terms of the problem data but may take any of several quantities in a problem and set up an equation by expressing that quantity in two mathematically independent ways, accounts for the superior power and flexibility of algebra over arithmetic.

Fractions as Whole Numbers

By S. L. Berman, *Stuyvesant High School, New York, N. Y.*

Ask a pupil (and most teachers for that matter) for an offhand answer to the simple question, "What is a fraction?" and the replies will not be characterized by their uniformity. True, most will define a fraction as "part of a number," but after that there will come a variety of answers varying with the maturity and understanding of the individual who answers. "A number with a numerator and a denominator"; "a number which is less than one"; "an indicated division"; "an indicated multiplication"; are some of the other definitions. No doubt each definition has its merit. Let us inspect and analyze.

A fraction *is* a part of a number. Yet that definition isn't limiting enough, and leads to difficulties when we begin to generalize. A nickel is part of a dollar. A quart is part of a gallon. An ounce is part of a pound. The number three is part of a score. Nevertheless, a nickel, or a quart, or an ounce, or the number three, is not considered a fraction normally. Or else, *every* number is a fraction; and the above definition of fraction ceases to have any significance. On the other hand, every fraction becomes a whole number. The youngster doesn't think of half an orange as a fraction, or a part, any more than he would think of an orange as a part of a dozen oranges. To him, any part, no matter how small, is *one* thing and two such parts are *two* things. Can we ask our youngsters to go through such seemingly paradoxical reasoning?

The answer, "a number with a numerator and a denominator," is a physical description of a fraction. On the surface, it shows no real understanding of the meaning. On being questioned, some will base their answers upon the familiar division of a number into parts. Rarely will one be found who can explain why the names—numerator and denominator—were given to these numbers. Few teachers, I am sorry to say, have ever bothered to find out that a numerator *counts* or *numbers*, while a denominator *names, classifies*, or *designates the units*. Still the above definition, worth while or not, often furnishes the youngster's only contact with the meaning of a fraction.

The definition, "a number less than one," brings forth some comment likely again to be thought facetious. It causes more and more definitions to harass the pupil; proper fraction, improper fraction, mixed number, etc. It seems to me that a pupil cannot think of something less than one. If he can see it and hold it, it is *one* to him. Or perhaps, he doesn't understand what is meant by "one." Furthermore, when he reaches "directed numbers" he meets again a number "less" than one in the form of the negative integer. The identification of the fraction by its definition becomes vague and confused in his mind.

The answer, "an indicated division," can be considered now. This is a non-popular but mathematical definition of a fraction. However, a fraction may be considered in two senses: the dynamic, or operational sense, and the static. Using the fraction 8/2 as an example, the given definition would say that 8 is to be divided by 2 and when that division is accomplished, the result is a fraction no longer. The number may be written just as well in the form $8 \div 2$. The above definition then recognizes a fraction only as a "shorter" way of writing a division. None of your "part of a number" idea is implied. Thus, the ratio 5:7 is also a division. (Note that the fraction line and the ratio sign both make up the earlier division sign.) This, in the case of the fraction 8/2, seems quite logical and understandable to the pupil. In the case of 5:7, or 5/7, however, the youngster begins to lose confidence in the definition. The operational phase is lost and the static character of a fraction assumes the domi-

nating position in his thought. He is satisfied when the operation of division is applied to cause 8/2 to become 4, but no operational device can cause 5/7 to lose its fractional character. The definition then is not completely satisfying. What is a fraction anyhow?

The last of the given definitions, namely "A fraction is an indicated multiplication," attempts to consider the static character of a fraction. Here, 8/2 is treated as 8 times $\frac{1}{2}$, and 5/7 is regarded as 5 times 1/7. This approaches most closely the meanings of numerator and denominator. However, the explantaion of the $\frac{1}{2}$ and the 1/7 is left to other definitions.

But perhaps we are on the right track. Suppose we consider 8 *times* $\frac{1}{2}$, as 8 halves; and 5 *times* 1/7, as 5 sevenths. In other words, we allow the fraction to take on the meaning of a whole number but in *units other than one's*. In fact an integer becomes a fraction whose units are one's. The key to understanding the fraction is the discussion of unit rather than the confusing definitions. Wouldn't it be simpler and far more worth while for teachers to treat fractions as whole numbers with due emphasis upon the theory of changing units?

The argument will be raised that it will be difficult to make a primary school youngster understand the idea of change of unit. As a matter of fact, the earliest introduction to numbers comes when the child learns to count, that is, *count by one's*. Then comes counting by *ten's*, or by *five's*, and then by *two's*. Whether or not he is aware of it, the *units* are being changed. In its initial stages, multiplication is extended addition, or counting, to the child. Thus, at first, 4 times 3 will be the sum of 4 units of 3 each. Actually, in performing this multiplication, he will be changing from units of threes to units of ones.

Similar teaching can extend this idea to deal with change of units to halves, quarters, etc. In giving the problem $9 \div 3$, doesn't the teacher ask, "How many threes are there in 9?" The actual form of division or multiplication is unchanged, but a new point of view is substituted for or supplements the previous one. The present method of adding or subtracting fractions is in few cases completely understood by the pupil. He goes through the motions in order to humor a teacher who has told him that "this is how we do it" by way of explanation. How much simpler it would be for him to begin addition of fractions (that is, combination by addition) by learning how to add coins of different denominations, or lengths of different units. When he learns to change quantities to be added to similar units, the addition of fractions will hold no terrors for him. Fractions will be merely other types of numbers whose units similarly have to be made to correspond to be added.

Will he have to reduce his answers to their "lowest terms"? It isn't necessary in all cases. The manufacturer will sell you 3 twelfths of a dozen, not $\frac{1}{4}$ of a dozen. The surveyor will measure the frontage of a house to be 30 feet, not 10 yards. The tailor will measure the length of a pair of trousers as 30 inches, not $2\frac{1}{2}$ feet. The pupil can tell most readily how many 10-cent War Stamps he can buy with his money by finding out how many units of ten cents each he has, not by the number of dollars. We still buy meat by the pound. In other countries, it is probable that the ounce or the gram is the unit. We should reduce numbers—any numbers—to appropriate units rather than to lowest terms.

There are many more implications than it is possible to discuss in an article of this nature. The minute details have to be left undiscussed. One more thought! The secondary school teacher of mathematics will say to himself, "All that is for the teacher of arithmetic. It's too late for me to do anything." He is partly right, but oh! so wrong. He might try to teach fractions as whole numbers with varying units to his algebra classes. Perhaps fewer pupils will add one eighth and one eighth and get two sixteenths; or subtract $2x$ from $3x$ and get 1; or add $\frac{1}{2}$ and $\frac{1}{3}$ and get—but who can tell?

◆ THE ART OF TEACHING ◆

The Language of Mathematics

By JOHN K. MOULTON
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EVERY body of knowledge has its own language, and most, at the same time, share a common language with the mother tongue. This is true of mathematics, just as it is of biology, art, or French; and it will be the purpose of this article to reflect a bit on the implications of this language area for the secondary school teacher of mathematics. Since there are so many more non-college preparatory students in secondary schools than there are college preparatory students, the point of view here is that of the teacher of the non-college preparatory students.

World War II has brought about a general awareness of the importance of mathematics to the average (non-college) person. This is apparent to any casual school visitor if he but observe the classes in "basic mathematics," "modern life" mathematics, "fundamental" mathematics, and the like. (The series is infinite.) The necessity of a knowledge of the fundamentals of mathematics (as differentiated from arithmetic) to the average student is something that mathematics teachers have long known. But as a group we have not always been sufficiently ingenious—or is it unbending?—in putting our ideas across to the average pupil and parent. The War has done that for us. And it is now our responsibility to keep alive these ideas in the average person's mind. This is, naturally, a complex and many-sided problem. It involves good guidance, a study of motivation, stimulating teaching aids, a realization of the air-age, and an understanding of the great range of pupil needs. And one small segment of our responsibility is that we become adept at the teaching of the language of mathematics.

Our *average* student had difficulty enough with the English tongue back in the days when it wasn't expected of him that he manipulate it in a mathematical setting. It is a troublesome thing, this native tongue of ours, full of unreasonable spellings, odd punctuations, complex structure, and unlimited elasticity of connotations. When forced into the rigidity of a mathematical setting, the English language becomes one additional hurdle whose location, height and method of clearing we must teach. In other words, in addition to teaching "mathematics," we must teach the "English of mathematics."

While there are countless ways of organizing the task of teaching the language of mathematics, these three areas of concern are submitted in a spirit of pragmatism:

1. Vocabulary
2. Reading for comprehension.
3. Formulating precise and accurate statements

Vocabulary, of course, is basic. Certainly few will deny that we need to know the meanings of the words we use. In learning a foreign tongue, one must learn its vocabulary. Most students accept this task as a matter of course. However, in studying mathematics, the student feels that he is on familiar terms with the language; he doesn't see that hurdle, and usually barks his shins on it. Are vertical angles really vertical? Or again, ask a class of beginning "basic" mathematics students, or begin-

ning geometry students for that matter, to define a circle. The collection of answers is either amusing or distressing, depending upon one's mood at the moment. The apparently simple question: "what is an angle?" brings forth much verbal floundering from the uninitiated; while the more advanced word "helix" is usually defined manually rather than verbally. Only the extraordinary student can describe a spiral staircase with his hands in his pockets.

The definition and use of the word "parallel" in mathematics is different from that in ordinary conversation; we have skew lines to think about, and Euclid's fifth postulate on our minds. In general it may be said that there are two kinds of words that should be taught in the mathematics vocabulary:

1. Words common in everyday English, whose meanings in a mathematical context are quite different (vertical, chord, degree, factor, base, similar, transposition, satisfy)
2. Words which are primarily mathematical. These are, on the whole, less troublesome, since their newness to the student does not lull him into a false sense of security (square root, radius, congruent, area, volume, pi, ratio, proportion)

The vocabulary of mathematics should be taught as scrupulously and conscientiously as the vocabulary of a foreign language is taught. How else is the mathematics student to know what we are talking about?

Reading for comprehension rests upon the knowledge of vocabulary. But comprehension requires more, it calls for the remembering of concepts in sequence, sometimes the formulation of the sequence itself; for a realization of the total idea. Teaching reading for comprehension in mathematics is not a task that we can, or should, place on the English department; it is rightfully a part of the teaching of mathematics. Teachers of English cannot be expected to have the requisite mathematical training. And reading for comprehension is not something that comes without effort; it, too, must be conscientiously taught. To find out how much of a statement his class comprehends the instructor has but to try a problem of this type: *it takes an airplane one hour and twenty minutes to fly from A to B, and eighty minutes to fly from B to A; how do you account for this?*

As anyone who has taught algebra or geometry can readily testify, lack of ability to read and comprehend is a major source of pupil error. The complete understanding of worded problems in algebra, the selection of the hypothesis and conclusion in a geometrical statement—how often we have seen students fail here, before they are challenged by the necessity of the mathematical processes. The same situation obtains, and perhaps to an even greater extent, with the non-college students. Here we are dealing with individuals of lesser reading ability, of more limited vocabulary; students who are usually not book-minded. True, we can utilize life experiences of these students, including other school subjects such as the manual and domestic arts, mechanical drawing and aeronautics, to put more meaning into their mathematics. But problems must be stated, as must the laws and principles of mathematics which solve the problems. Even in such microcosmic matters as deciphering an electricity bill, or filling out an income tax blank, the ability to read, comprehend, and to "figure" is required. There is no complete escape from the printed page.

Nor is it sufficient in our mathematics classes to have one unit on reading for comprehension and stop there; it is a non-ending task. Probably the time would be gainfully spent if, on every lesson involving worded problems, students were required to list two groups of facts for each problem: 1) what facts are given in this problem, and 2) what do we have to find? Surely if the latter question were fully comprehended the answers would not be given in square feet when square yards are required. And no student can solve a prob-

em correctly unless he comprehends the given facts. We teachers of mathematics need to accept fully the responsibility of teaching reading for comprehension in the area of mathematics.

Formulating precise and accurate statements is a "by-product" of mathematical training (though perhaps not all who read this article will agree). Mathematics teachers have no sole claim on this, any more than we do on vocabulary or reading for comprehension. Nevertheless, precision and accuracy of statements are in the spirit of our mathematical language; we cannot afford to neglect them in our instruction. Probably the most frequent contact the non-college student has with concise statements in his mathematical experiences comes with his use of formula. All common formulae, such as: $A=\pi r^2$, $A=\frac{1}{2}h(b+b')$, $d=rt$, and $c=prt$ represent precision and accuracy of statement found only in the exact sciences. And such statements are within the experience and ability range of our non-college students. Here again, knowledge of vocabulary is essential. It hardly needs be mentioned that students using the formula $A=\pi r^2$, for example, should know exactly what A, π, and r stand for.

The most compelling reason that we, as teachers of mathematics, should be concerned with precision and accuracy of statement is that the method of expression reveals the method of thinking. If we expect precise and accurate thinking, then we must stress precise and accurate expression. We should constantly strive for such method of expression in both the oral and written work of our students.

Notes from an Elementary Algebra Class

Even the most elementary presentations of the graph of a quadratic function point out that the "x" intercepts of the curve, distinct or coincident, are the roots of the function; and further that if the intercepts are not real, then the roots are not real. It does not take a particularly brilliant student to ask immediately just what relation the graph has to the roots in this case, and so far as I know, the texts are silent.

In the case of non-real, or complex, roots we know for the function $f(x)=ax^2+bx+c$, that $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ or $\dfrac{-b\pm\sqrt{4ac-b^2}}{2a}$ in which the real part, $-b/2a$, is merely the abscissa of the vertex; and in which we seek the graphic significance of the quantity, $\dfrac{\sqrt{4ac-b^2}}{2a}$. The ordinate of our vertex is $f\left(\dfrac{-b}{2a}\right)=\dfrac{4ac-b^2}{4a}$. Consider now the width of our parabola corresponding to an ordinate twice that of the vertex, i.e., put $\dfrac{4ac-b^2}{2a}=ax^2+bx+c$ and solve for the width of the parabola, which is the difference of the solutions of this equation, reducing readily to $\dfrac{\sqrt{4ac-b^2}}{a}$.

From this it appears that the semi-width of the parabola at this ordinate is the coefficient of the imaginary term of our complex root. Obviously if we reflect the parabola over its tangent at the vertex, this coefficient of the imaginary term is half of the portion of the x-axis intercepted by this reflected parabola.

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EDITORIAL

What Can the National Council Accomplish in the Next Twenty-Five Years?

HAVING just completed the first twenty-five years of its existence, and having made some progress in improving mathematical education in this country, it is only natural that we should inquire as to what may be accomplished in the next twenty-five years.

For one thing let us hope that more teachers of mathematics will discover that there is such an organization to which they should belong, that it publishes an official journal eight times a year which they might read with profit, and that it has also published a most outstanding series of yearbooks in the secondary field. These yearbooks are of inestimable value to mathematics teachers who should certainly read them if they do not buy them for their individual libraries.

There is a large group of teachers both in the rural and in the urban districts of the United States who do not know anything about the National Council, or its work. This holds true from the elementary school through the junior college. One great source of improvement will consist in the National Council's ability to locate such teachers and secure their interest. How can this best be done? The October (1945) issue of THE MATHEMATICS TEACHER was sent to a selected list of 2500 teachers of mathematics throughout the country whom the State Representatives of the Council thought were good prospects. In spite of the fact that the October number seemed to be well received, very few new subscriptions came in. We are up against a situation in this country that is difficult to handle. Too many mathematics teachers are not interested even when they know about the work of the Council.

Besides, the need of securing greater interest in the work of the Council among teachers of mathematics, the Council itself should find ways of helping teachers to solve some of the more perplexing problems that confront them today. We can mention only a few.

1. The great mass of pupils in our schools differ greatly in native ability, in interests, and in experience. How can such pupils be taught the same kind of mathematics without great injustice to all concerned? In fact, how can they be successfully taught together at all in the best sense of the word? We need to develop at least a two track system as far as content is concerned.

2. In spite of the need for reorganization of the content in mathematics, most schools continue to teach the traditional content to the detriment of all concerned. Something must be done to reorganize the work and make it practical for large numbers of pupils in the schools who cannot profit by studying the more academic and deductive phases of the subject. Such pupils will not go to college and therefore need something different from college entrance material.

3. For the most part algebra is taught on the mechanical level and geometry on the memoriter level, with the result that many pupils dislike both subjects. If more emphasis were placed on meaning and understanding and less on the mechanical aspects, pupils might obtain more value from their study of mathematics. We have made some progress in arithmetic, less in algebra, and almost nothing in this respect in geometry. The time is ripe for action, and the National Council is the organization that should lead the way.

Perhaps we can get some contributions

from teachers in the field as to how progress can be made, but the public and particularly many of the educationists will not stand idly by and wait for things to happen.

What we need most right now is more members, more devotion and interest in the improvement of mathematical education from the members we already have, and better cooperation of teachers on all levels in trying to solve some of the most important problems.—W.D.R.

National Council Election Postponed!

DUE to the fact that the February 1946 issue of THE MATHEMATICS TEACHER containing the official ballots (p. 92) of The National Council of Teachers of Mathematics was late in getting to many members of the Council, the Board of Directors at the annual meeting on February 22, 1946 in Cleveland, voted to extend the time of voting to April 15, 1946 and instructed Secretary E. W. Schreiber of the Western Illinois State Teachers College at Macomb, Ill. to send out postcard ballots to all members. Members should therefore mark either the postcard ballot or the one on page 92 of the February issue of THE MATHEMATICS TEACHER and mail it to Secretary Schreiber so as to reach him before April 15. It is the duty of every member to vote. The results of the election will be given in the May issue of THE MATHEMATICS TEACHER.—W. D. R.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR
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6. Nyberg, Joseph A., "Notes from a Class Room Teacher" (continued), pp. 82–85.
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11. Miscellaneous Notes.
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THE MATHEMATICS TEACHER
525 W. 120th Street, New York, N.Y.

NEWS NOTES

PROGRAM OF PUBLIC LECTURES
of the
SOCIETY OF FRIENDS OF SCRIPTA MATHEMATICA
and of the
YESHIVA INSTITUTE OF MATHEMATICS
1945–1946

1. Why has the Syllogism 19 Moods?—by Professor William Pepperell Montague, of Columbia University—Nov. 8, 1945.
2. Mapping the World—by Professor Edward Kasner, of Columbia University—Dec. 6, 1945.
3. Art and Mathematics—by Rutherford Boyd, January 9, 1946.
4. A Popular Exposition of the Theory of Relativity—by Professor Vladimir Karapetoff, of Cornell University, February, 1945.
5. New Light from Old Lamps—by Professor Charles Louis Karpinski, March, 1946.
6. Polygons and their Transformations—by Professor Jesse Douglas, of Brooklun College, April, 1946.

7–8. Lectures by Count Alfred Korzybski, (Titles and Dates to be announced.)

9–10. Problems and Methods in Modern Mathematics—by Professor Abraham A. Frankel of the University of Jerusalem—Dates to be announced.

11. Mathematics in the Talmud—by Professor Jekuthiel Ginsburg, Yeshiva College, New York.

The following review of the Eighteenth Yearbook of the National Council of Teachers of Mathematics on "Multi-Sensory Aids in the Teaching of Mathematics" appeared in the University of Michigan School of Education Bulletin for October 1945. We think it will be of interest to our readers. The yearbook can be obtained postpaid from the Bureau of Publications. Teachers College, 525 West 120 Street, New York 27, New York, for $2.00.—EDITOR

In comparison with aids in some of the other subjects, there seems often to be a bleak paucity of illustrative materials in mathematics classrooms. This fine yearbook, prepared by a committee under the chairmanship of Professor E. H. C. Hildebrandt of Northwestern University, shows what can be used and how to use it. The many articles cover a wide range of topics and the book, appropriately, is well illustrated. There are extensive lists of sources. Principals and superintendents may well secure the book and encourage mathematics teachers to enrich the illustrative aspects of their teaching.

Henry Holt & Company, Publishers, 257 Fourth Avenue, New York, N. Y., are printing copies about 14×20 inches on firm, but not stiff, blue paper, of "The River Mathematics," which appeared as the frontispiece in the October (1945) issue of THE MATHEMATICS TEACHER.

Anyone wishing copies of this picture may secure same by sending 10¢, to cover the mailing tubes and postage cost, to the above address.—EDITOR

The Rhode Island Mathematics Teacher's Association held its Fall Meeting with the Rhode Island Institute of Instruction at the Rhode Island College of Education at Providence on Thursday, October 25, 1945 at 2 P.M. Professor A. Harry Wheeler of Worcester, Mass. gave an address on "Two-dimensional and Three-dimensional Models for Geometry." Mr. M. L. Herman of the Moses Brown School in Providence is president and Miss Harriet C. Whitaker of the Lincoln School in Providence is secretary-treasurer.

MATHEMATICS SECTION OF THE COLORADO EDUCATION ASSOCIATION
MATHEMATICS

President—Kenneth E. Gorseline, Denver.
Vice-President—A. E. Mallory, Greeley.
Secretary—Mary C. Doremus, Denver.

FRIDAY, OCTOBER 26

10:00 A.M.—Capital Life Insurance Company Auditorium, 16th Avenue and Sherman Street.

Business meeting.

A discussion of the "Second Report of the Commission on Post War Mathematics," Dr. Harl R. Douglass, Director of College of Education, University of Colorado, Boulder.

12:15 P.M.—Luncheon, Blue Parrot Inn.

"The Mathematics Rural High School Pupils Must Know," Mr. Harry W. Charlesworth, East Denver High School.

The report of the nominating committee was as follows

Vice President: Dr. Aubrey Kemper
Secretary-treasurer: Mr. Ernest R. Bails
Board Members:
Mr. Russell Casement
Mrs. Leota Hayward
Miss Ruth Hoffman
Miss Helen Taylor

The discussion of the "Second Report of the Commission on Post War Mathematics" conducted by Dr. Harl R. Douglass created a lively discussion with many members from the audience participating. Dr. Douglass presented the high lights of the report for forty minutes; then allowed forty minutes for discussion. At the end of the discussion, several comments were going around, such as "It's about time our text books line up with the principles set forth in the *Second Report.*"

The Duodecimal Society of America has just announced the conferring of its Annual Award for 1944 on F. Emerson Andrews, Tenafly, N. J., and of the Award for 1945 on George S. Terry, Hingham, Mass.

The Duodecimal Society is engaged in educating the public in counting by twelves, instead of tens, and in the application of the 12-Base in mathematics, weights and measures, and other branches of pure and applied science. Its Annual Awards are conferred on those who have made outstanding contributions to the development of duodecimals, and to the advancement of the purposes of the Society.

Mr. Andrews is Publication Manager for the Russell Sage Foundation. He received his degree in Arts from Franklin and Marshall College, and has written a number of magazine articles on duodecimals, in addition to the book, "New Numbers," which is now in its second edition.

Mr. George S. Terry received his engineering degree in Turbo Electrics from the University of London. He is the author of the "Duodecimal Arithmetic," with its extensive mathematical tables for duodecimals. Among his other works are "The Dozen System," and many papers on duodecimals as applied to the Theory of Numbers.

1946 SUMMER SESSION AT TEACHERS COLLEGE, COLUMBIA UNIVERSITY IN THE TEACHING OF MATHEMATICS

Teachers College, Columbia University, will offer the following courses in the teaching of mathematics in the summer session of 1946, which begins on July 8, and ends on August 16.

By Professor Harold Fawcett: A one-week course on the teaching of geometry. (July 15 to 19. No credit, $10 fee.)

By Professor Howard Fehr: Teaching algebra in secondary schools; professionalized subject matter in advanced secondary mathematics.

By Dr. Nathan Lazar: Teaching geometry in secondary schools; history of mathematics.

By Mr. Gordon Mirick: Elementary mechanics (statics); observation and participation in the teaching of geometry.

By Professor W. D. Reeve: Teaching and supervision of mathematics—junior high school; teaching and supervision of mathematics—senior high school.

By Professor W. S. Schlauch: Modern business arithmetic.

By Professor Carl N. Shuster: Teaching arithmetic in elementary school; field work in mathematics.

There will be held during the summer session, on consecutive Thursdays beginning on July 11, five special lectures and discussions in which all of the instructors above and other persons from outside will participate for the purpose of bringing before the students vital questions relating to the reorganization and teaching of mathematics in the post-war world. There will be an opportunity for discussion in which all of the students will be invited to take part. These conferences have come to be a common meeting place for all students, instructors, and guests, and thus serve both professional and social ends. Registration for these special lectures and discussions is not necessary and all those who are interested in the improvement of mathematical education are invited to attend.

THE MATHEMATICS TEACHER will be glad to print programs for teachers of mathematics to be given in the summer session of other schools like the above provided they are sent to the editor in time for a particular issue.

The Iowa Association of Mathematics Teachers met in Des Moines on Friday, February 1, 1946.

The 1944–45 officers of the Association and who had charge of the program, were as follows:

President: Miss Ethel Cain, Roosevelt High School, Des Moines
Vice-Pres.: Miss Esther Row, Perry High School, Perry
Sec.-Treas.: H. Vernon Price, University High School, Iowa City

Officers elected for the 1945–1946 year include the following:

President: Miss Esther Row, Perry High School, Perry
Vice-Pres.: Miss Dora Kearney, Campus High School, Cedar Falls
Chairman, Advisory Board: Dr. H. Van Engen, Iowa State Teachers College, Cedar Falls
Elected to Advisory Board:
(for one year)
Dr. E. W. Chittenden, State University of Iowa, Iowa City
Miss Louise Gaekle, Muscatine Junior College, Muscatine
(for three years)
Miss Freda Blum, Fort Dodge Junior High School, Fort Dodge
Miss Esther Quimby, Algona High School, Algona
Dr. Harold Trimble, Iowa State Teachers College, Cedar Falls
(The Secretary-Treasurer carried over on a three-year term.)

The program of the meeting was as follows:

PROGRAM

Iowa Association of Mathematics Teachers
St. John's Lutheran Church
Des Moines, Iowa
Friday, February 1, 1946

Luncheon Meeting at 12:15 P.M.
Greetings from Dr. Maurice Hartung, University of Chicago
Report on Post-War Planning Commission, Dr. H. Vernon Price, University of Iowa
Display Materials, Miss Ruth Miller, Ames

Afternoon Meeting at 2:00 P.M.
Business meeting
The Actuarial Profession—Mr. William Rae, Actuary, Bankers Life Insurance Company, Des Moines
A Challenge to Teachers of Mathematics—Dr. Maurice Hartung, University of Chicago
Report of Committee on Revision of Constitution—Miss Viola Smith, Waterloo

BOOK REVIEWS

The Vertical Line Method in Mathematics, the Easy Way in Arithmetic. By Laura M. Stephens, George W. Brackenridge High School, San Antonio, Texas.

This brief pamphlet describes a pattern for the multiplication and division of fractions which has been successfully used in the San Antonio High Schools. Mixed fractions are converted to improper fractions, a long vertical line is drawn. In multiplication, the numerators are written on the right of the line, the denominators on the left; they are cancelled, multiplied, and reduced. In division, the form is the same, save that the terms of the divisor are inverted right to left rather than up and down.

Eleven examples are given in the pamphlet, which explains "the vertical line method may be used to advantage in solving problems in higher matnematics and in science when both multiplication and division are necessary. The need of scratch paper or of side figuring is eliminated by this method and correct results are obtained quickly."

Miss Stephens has made analyses and tests concerning this method, the results of which are, however, not given in this pamphlet. Her method saves space and number of "strokes" for it reduces copying of numbers and eliminates need for the $\times$, $\div$ and $=$ symbols. At the end of a semester of commercial arithmetic in which both methods had been used, she gave two 10 minute tests to each half of a class; so that each group of 16 would be tested by both methods. In Test 1 of the 77 exercises worked by the "text" method 33 were correct, and of the 88 worked by the vertical method 44 were correct. In Test 2, 93 out of 129 exercises were correct by the textbook method, while 110 of 136 done by the vertical line method were without error. Hence it is apparent that the vertical line method gave this class more speed and accuracy than the other. The exercises given were apparently similar to $72\frac{2}{3} \times 57\frac{3}{8}$ and $35\frac{5}{8} \div 32\ 11/12$.

The pattern is not concerned with promoting meaningful understanding of fractions problems. But any teacher who feels the need of a concise pattern for solving exercises or problems of this type should be interested in Miss Stephens' work.

HOLMES BOYNTON

The College and Teacher Education. Prepared for the Commission on Teacher Education of the American Council on Education by W. Earl Armstrong, Ernest V. Hollis, and Helen E. Davis. American Council on Education, 744 Jackson Place, Washington 6, D. C. 311 pages. Price $2.50.

This book seeks to present a comprehensive view of the task of teacher education, with special emphasis on the interrelatedness of all aspects of the matter and on the resulting importance of group methods. In so doing it tries to convey the characteristic drive of the Commission's nationwide cooperative study. No attempt has been made to bring into focus all of the current issues in teacher education, nor to draw extensively on educational research and literature, nor yet to make use of all information from the participating institutions on any one problem. Rather, data have been somewhat carefully selected for their value in pointing up and making vivid the discussion of those topics which the authors have judged to be of widest interest. The accent is on the grass roots—on what has, and what has not, worked in terms of quite specific conditions and actual situations.

After an introductory chapter on the origin of the Commission on Teacher Education and its program, the authors present and interpret what the associated colleges and universities did toward implementing programs of student personnel, general education, specialization in major fields, and professional education. They then devote a chapter to the recurring emphases of these several programs and provide additional material on the most important of them, student teaching. The remaining three chapters are given to joint efforts of school systems and colleges for the in-serivce education of teachers, to the subject of integration and the group approach, and to the authors' own final reflections and conclusions.

The twenty institutions of higher learning that took part in the cooperative study of teacher education included universities, colleges of liberal arts, and teachers colleges. Some of them are coeducational, others admit only men or women students. Some are privately endowed, others are publicly supported. In some, teacher education is the sole institutional objective, in others it is the foremost among several interests, and in others the concern of only a specialized minority. Every major region and type of community in the United States, with all the cultural differences they imply, are likewise to be found in the several campus environments. In short, no two of the cooperating centers were exactly alike and practically the full range of diversity in American higher education was represented.